INTRODUCTION

PERCENT ERROR AND PERCENT DIFFERENCE

A <u>percent error</u> should be calculated when an experimental value E is compared to a standard or accepted value S of the same quantity. We express the difference between E and S as a percent of the standard value S:

$$PE = \frac{E - S}{S} \times 100$$

(1)

This formula yields a positive result if the experimental value is high in comparison to the standard value, and a negative result if it is low.

A <u>percent difference</u> should be calculated when two experimental values, E_1 and E_2, are compared to each other, and there is no standard value for comparison. In this case we express the absolute value of the difference between the experimental values as a percent of the average of the two values:

$$PD = \left| \frac{E_1 - E_2}{\frac{1}{2}(E_1 + E_2)} \right| \times 100 = \left| \frac{E_1 - E_2}{(E_1 + E_2)} \right| \times 200$$

(2)

Percent difference is always positive.

THE METHOD OF DIFFERENCES

The Method of Differences is used when some variable is believed to change by equal amounts in successive measurements. This method yields the average change in the variable per interval.

A physical example of such a case is the stretching of a spring by a force which increases by equal amounts in successive intervals. Another example is the speed of a dense falling body measured at equal time intervals.

As a concrete example to illustrate the Method of Differences, suppose that we want to measure the width of one of a number of identical floor tiles in a room.

One method is to place a measuring rod down, measure a tile, move the rod, measure another tile, and so on. We could then find the average of the individual measurements. However, moving the rod increases the experimental error.

A more precise method is to place the rod down only once, and then to take a set of readings of the positions of successive cracks (assumed to be of negligible thickness). For example, six successive readings would span five tiles. Let the six readings be a, b, c, d, e and f.

There are several ways in which these six numbers could be combined in order to yield a single tile width.

A poor method would be to use the formula

$$w = \frac{f - a}{5}$$ (3)

The result would be approximately correct, but less precise than it could be, since we have used only two of the six readings.

A method that looks better at first is to calculate the five widths, b-a, c-b, d-c, e-d, and f-e, and then average them. The equation for this procedure is

$$w = \frac{(b - a) + (c - b) + (d - c) + (e - d) + (f - e)}{5}$$ (4)

Close inspection of this equation, however, shows that it reduces to equation (3), so we have gained nothing by all of our extra work.

The Method of Differences uses each of the six readings once, and no reading cancels out. In order to use it, we must have an even number of readings. For an odd number of readings, either the first or last reading must be discarded. Then, we divide the readings into two sets. In our example, set one would consist of readings a, b and c; set two would consist of readings d, e and f. We can get one estimate of the average by using readings a and d,

$$\frac{d - a}{3} .$$

Here, the distance (d-a) spans three tiles, so we have divided by three. We can get two other estimates using the pairs b and e and c and f,

$$\frac{e - b}{3} \quad \text{and} \quad \frac{f - c}{3} .$$

The method of differences uses the average of these three estimates

$$w = \frac{\left(\frac{d - a}{3}\right) + \left(\frac{e - b}{3}\right) + \left(\frac{f - c}{3}\right)}{3} = \frac{(d - a) + (e - b) + (f - c)}{3^2}$$ (5)

In the general case, suppose that we have 2n successive readings, A_1 to A_n and B_1 to B_n, of some variable S. Then the average change in S per interval is given by

$$\Delta S = \frac{(B_1 - A_1) + (B_2 - A_2) + \cdots + (B_n - A_n)}{n^2}$$ (6)

UNCERTAINTY IN DATA AND CALCULATIONS

I. Limitations on Precision of Measurement

No measurement is exact. The precision of a measurement is limited by the nature of the measuring instrument itself, by the conditions of measurement, and by the skill of the person using the instrument.

For example, the precision of our measurement of the length of a board, using a meter stick, is limited by the fact that the smallest division on the stick is a millimeter. We can estimate the length of the board to a precision of the nearest fifth or tenth of a millimeter by mentally dividing the millimeter into smaller divisions, but soon we reach a limit. Even if we were to use a magnifying glass in an effort to divide the millimeter into even smaller parts, we would eventually be limited by irregularities in the markings on the meter stick.

A common source of error in using a meter stick is parallax error. This is error caused by the line of sight not being precisely perpendicular to the stick. For example, measuring the length of a sheet of paper by placing the stick down flat on the paper easily leads to parallax error; it would be better to place the meter stick on edge so that the markings touch the paper.

Parallax error can occur in many situations; it occurs, for example, in reading an electrical meter when the pointer moves in a different plane from the scale.

II. Absolute and Relative Uncertainty

Suppose that we measure the length of a piece of paper as 20.00 cm, by using a meter stick. Then, after considering the process of measurement, we decide that we might be in error by as much as 0.02 cm (one-fifth of the smallest division of the stick). We usually express this by saying that the length is 20.00 ± 0.02 cm.

In this case, 0.02 cm would be the <u>absolute uncertainty</u> in the measurement or the <u>possible absolute error</u> in the measurement.

We define the <u>relative uncertainty</u> in a measurement or the <u>possible relative error</u> in a measurement to be the ratio of the absolute uncertainty to the actual measurement; for our example, 0.02 cm/20.00 cm = 0.001 = 0.1 %.

Note that the absolute uncertainty has the same units as the measurement; whereas the relative uncertainty is unitless and is often expressed as a percent.

Uncertainty and possible error are synonyms. Uncertainty is more often associated with measurements. Possible error is more often associated with results calculated from measurements.

III. Use of Differentials to Represent Uncertainties

Let the length of the paper in section II be called L. Then we represent the absolute uncertainty in L by the symbol dL. We represent the relative uncertainty in L by the ratio dL/L.

Since relative uncertainties are usually rather small, the concept of a differential is useful in determining how two or more uncertainties combine together when measurements are used in mathematical operations. This will be illustrated in the next section.

IV. How Uncertainties in Data Affect Calculated Results

Whenever a calculation is performed using measured data, there will be uncertainty in the result. We will now consider how to determine the uncertainty in a result, when data is added, subtracted, multiplied, divided, or raised to an integer power.

A. Addition

Let S be the sum of two independent[†] quantities, S = A + B.

The "worst-case" scenario is that both A and B were measured small or both were measured large; therefore, dS = dA + dB.

Example: (25.0 cm $\pm$ 0.2 cm) + (10.0 cm $\pm$ 0.5 cm) = 35.0 cm $\pm$ 0.7 cm

[†]For calculations involving dependent quantities, refer to section IV-F.

B. Subtraction

Let D be the difference of two independent quantities, D = A - B.

The "worst-case" scenario is now that A was measured too large and B too small or vice versa; therefore, dD = dA + dB.

Example: (25.0 cm $\pm$ 0.2 cm) - (10.0 cm $\pm$ 0.5 cm) = 15.0 cm $\pm$ 0.7 cm

C. Multiplication

Let P be the product of two independent quantities, P = AB.

The "worst-case" scenario is that both A and B were measured small or both were measured large. Since we are multiplying, we are concerned not with the absolute uncertainty of A or B, but with the relative uncertainties.

The result is dP/|P| = dA/|A| + dB/|B|.

Example: (5.00 m $\pm$ 1%)(8.00 m $\pm$ 3%) = 40.0 m² $\pm$ 4% = 40.0 m² $\pm$ 1.6 m²
(1.6 m² is obtained by calculating 4% of 40.0 m².)

This rule can be verified by looking at the product of the largest values for A and B.
          (5.00 m + 1%)(8.00 m + 3%) = (5.05 m)(8.24 m) = 41.6 m²

D. Division

Let Q be the quotient of two independent quantities, Q = A/B.

The "worst-case" scenario is now that A was measured too large and B too small or vice versa. Since we are dividing, we are concerned not with the absolute uncertainty of A or B, but with the relative uncertainties.

The result is dQ/|Q| = dA/|A| + dB/|B|.

Example: (3.00 g ± 1%)/(1.50 cm³ ± 3%)=2.00 g/cm³ ±4% = 2.00 g/cm³ ± 0.08 g/cm³

This rule can be verified by looking at the product of the largest value for A and the smallest value for B.

$$(3.00 \text{ g} + 1\%)/(1.50 \text{ cm}^3 - 3\%) = (3.03 \text{ g})/(1.455 \text{ cm}^3) = 2.08 \text{ g/cm}^3$$

Note: The general rule for addition and subtraction is that the absolute uncertainty in a sum or difference is equal to the sum of the absolute uncertainties in the quantities added. The general rule for multiplication and division is that the relative uncertainty in a product or quotient is equal to the sum of the relative uncertainties in the factors.

Simply put: If F = A ± B, dF = dA + dB.

If G = A × B or G = A/B, then dG/|G| = dA/|A| + dB/|B|.

E. Expressions Containing Only Powers, Multiplications and Divisions

Since powers are simply repeated multiplication, we have, in effect only multiplications and divisions. From the results of parts C and D, we see that we can simply sum the relative errors.

For example, if A = B^{p}C^q/D^r, then dA/|A| = p dB/|B| + q dC/|C| + r dD/|D|

The density of a cone is $\rho = \dfrac{m}{V} = \dfrac{m}{\frac{1}{3}\pi r^2 h}$, where m is the mass, r is the radius, and h

is the height. The relative uncertainty in the density is dρ/ρ = dm/m + 2 dr/r + dh/h, where dm, dr and dh are the absolute uncertainties in the mass, radius and height respectively.

[†]F. Expressions containing dependent quantities.

One must be careful to only apply the rules above to expressions containing independent quantities or else errors can be double or triple counted. Two quantities calculated from the same measurement are dependent, as are a measured quantity and a value calculated from it.

One simple geometric example of the latter case is the calculation of the difference between the circumference and diameter of a circle. Consider a circle with a diameter measured to be 10.0 cm ± 0.2 cm (10.0 cm ± 2 %). Its circumference is C = πd. Since π is a constant, its relative uncertainty is zero and the circumference will have the same relative uncertainty as the diameter, 2 %. The result is C = 31.4 cm ±

0.6 cm. Using the rules above, we would find $C - d = 21.4$ cm ± 0.8 cm. The uncertainty in the difference is actually less than 0.8 cm. To calculate the actual value we must put together the two formulas above, $C - d = \pi d - d = (\pi-1)d$. Since $(\pi-1)$ is a constant, its relative uncertainty is zero and the difference will have the same relative uncertainty as the diameter, 2 %. Thus, $C - d = 21.4$ cm ± 0.4 cm.

An example of the first case would be the ratio of the volume of a sphere to its surface area. Consider a sphere with a diameter measured to be 10.0 cm ± 0.2 cm (10.0 cm ± 2 %). The surface area of the sphere is $A = \pi d^2$. Since π is a constant, its relative uncertainty is zero and the surface area will have twice the relative uncertainty of the diameter, 4 %. The result is $A = 314$ cm ± 4 %. The volume of the sphere is $V = 1/6\, \pi d^3$. Since $\pi/6$ is a constant, its relative uncertainty is zero and the volume will have three times the relative uncertainty of the diameter, 6 %. The result is $V = 524$ cm ± 6 %. Using the rules above, we would find $V/A = 1.66$ cm $\pm$ 10 %. The relative uncertainty in the ratio is actually less than 10 %. To calculate the actual value we must put together the formulas above, $V/A = (1/6\, \pi d^3)/(\pi d^2) = d/6$. Since 6 is a constant, its relative uncertainty is zero and the ratio will have the same relative uncertainty as the diameter, 2 %. Thus, $V/A = 1.66$ cm ± 2 %.

As shown in the two examples, when an expression contains dependent quantities, it is necessary to algebraically simplify the expression before calculating the relative uncertainty.

G. Complex Expressions

Complex expressions may be handled by repeated use of the previous results.

Example: Let $A = \dfrac{BC^2}{DE - F}$ where $B = 50.0 \pm 0.5\ \frac{m^3}{kg\,s^2}$, $C = 10.0 \pm 0.2$ kg,

$D = 100 \pm 1$ m, $E = 3.00 \pm 0.06$ m, and $F = 50.0 \pm 1.0$ m².

Solution: $A = \dfrac{BC^2}{DE - F} = \dfrac{\left(50.0 \pm 0.5\,\frac{m^3}{kg\,s^2}\right)\left(10.0 \pm 0.2\,kg\right)^2}{\left(100 \pm 1\,m\right)\left(3.00 \pm 0.06\,m\right) - \left(50.0 \pm 1.0\,m^2\right)}$

$A = \dfrac{\left(50.0\,\frac{m^3}{kg\,s^2} \pm 1\%\right)\left(10.0\,kg \pm 2\%\right)^2}{\left(100\,m \pm 1\%\right)\left(3.00\,m \pm 2\%\right) - \left(50.0 \pm 1.0\,m^2\right)}$

$A = \dfrac{\left(50.0\,\frac{m^3}{kg\,s^2} \pm 1\%\right)\left(10.0\,kg \pm 2\%\right)^2}{\left(300\,m^2 \pm 3\%\right) - \left(50.0 \pm 1.0\,m^2\right)} = \dfrac{\left(50.0\,\frac{m^3}{kg\,s^2} \pm 1\%\right)\left(10.0\,kg \pm 2\%\right)^2}{\left(300\,m^2 \pm 9.0\,m^2\right) - \left(50.0 \pm 1.0\,m^2\right)}$

$A = \dfrac{\left(50.0\,\frac{m^3}{kg\,s^2} \pm 1\%\right)\left(10.0\,kg \pm 2\%\right)^2}{\left(250\,m^2 \pm 10\,m^2\right)} = \dfrac{\left(50.0\,\frac{m^3}{kg\,s^2} \pm 1\%\right)\left(10.0\,kg \pm 2\%\right)^2}{\left(250\,m^2 \pm 4\%\right)}$

$A = 20.0\,\frac{kg\,m}{s^2} \pm 9\% = 20.0\,\frac{kg\,m}{s^2} \pm 1.8\,\frac{kg\,m}{s^2}$

SIGNIFICANT FIGURES AND DECIMAL PLACES

Often, we wish to quickly estimate the precision of our calculated results without applying the rigor of the previous section. To do this, we use rules that relate the number of decimal places or significant figures we can keep in a calculated value to the number of decimal places or significant figures in our data.

I. Definitions

The number of decimal places in a number is the number of digits to the right of the decimal point.

The number of significant figures in a number is the total number of digits, exclusive of leading zeros.

II. Examples

The following table shows the number of decimal places and the number of significant figures in five numbers.

Number	Number of Decimal Places	Number of Significant Figures
15.73	2	4
0.0072	4	2
200.6	1	4
1.2700	4	5
4300	0	2, 3 or 4

The ambiguity in the number of significant figures in the last example is easily removed by using scientific notation. 4.30×10^3 is three significant figures.

III. Rules for Rounding Off Calculated Results

In addition or subtraction, keep as many <u>decimal places</u> in the result as the <u>smallest</u> number of decimal places found in any of the numbers being added or subtracted.

Examples: 20.5 + 1.483 = 22.0

19.03 - 18.96 = 0.07

10.512 - 9.8 = 0.7

4.93 + 6.26 = 11.19

Notice that the number of significant figures in the result can be more than the number of significant figures in either number or less than the number of significant figures in either number.

In multiplication or division, keep as many <u>significant figures</u> as the <u>smallest</u> number of significant figures found in any of the numbers being multiplied or divided.

Examples: $38.75(49.186)/1.48 = 1.29 \times 10^3$ or 1290, but NOT 1287.81.

$(1.237)(43.9)^2 = (1.237)\ (43.9)\ (43.9) = 2380$ or 2.38×10^3

The reasonableness of these rules may be established by using the methods discussed in the previous section.

RESISTOR COLOR CODES

The nominal resistance of a resistor is specified by the colors of three bands printed on the resistor. If present, a fourth band indicates the tolerance and a fifth band gives a reliability rating. When only three bands are present the tolerance is understood to be ± 20 %.

Band 1 is the band nearest an end of the resistor.

Bands 1 and 2 specify the first two digits of the resistance, A and B, respectively. Band 3 specifies the exponent C in the formula $R = AB \times 10^C$. A, B and C have integer values found in the table below. Note that A and B are not multiplied, but are the digits of a two digit number with A in the tens place and B in the units place (see examples below).

Color	silver	gold	black	brown	red	orange	yellow	green	blue	violet	grey	white
Band 1 or 2			0	1	2	3	4	5	6	7	8	9
Band 3	-2	-1	0	1	2	3	4	5	6	7	8	9
Band 4	10 %	5 %		1 %	2 %			0.5 %	0.25 %	0.1 %	0.05%	

Examples:

Band 1	Band 2	Band 3	Band 4	A	B	C	Tolerance	Resistance
brown	black	brown	silver	1	0	1	10 %	100 Ω ± 10 %
red	red	orange		2	2	3	20 %	22 kΩ ± 20 %
violet	yellow	gold	gold	7	4	-1		7.4 Ω ± 5 %

Note: Most resistors with a tolerance of 2% or less have *four* bands that specify the nominal value. We will not encounter these in this class.