Vector Addition

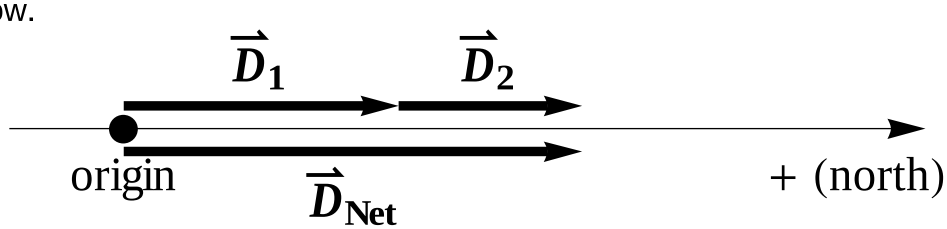
We have previously shown how to find the components of a vector from its magnitude and direction. We have also shown how to find the magnitude and direction of a vector, given its components. We will use that skill here in one method for vector addition.

In experiments you have looked at addition of displacement vectors in one and two dimensions. You have also solved problems involving vector addition of displacement vectors in one dimension.

One dimensional vector addition: In the case of one-dimensional vectors, the direction of the vector is frequently indicated by a plus or minus sign. For example, north could be positive and south could be negative, or up could be positive and down could be negative, or down the hallway toward the Science Center could be positive and the opposite direction down the hallway could be negative.

Suppose you walk 1.20 m north and then an additional 0.80 m north. If we choose north to be the positive direction, then your net displacement would be $\vec{\mathbf{D}}_{Net} = \vec{\mathbf{D}}_1 + \vec{\mathbf{D}}_2$, where $\vec{\mathbf{D}}_1$ = +1.20 m ("+" refers to the positive direction, in this case north). The net displacement is (+1.20 m) + (+0.80 m) = +2.00 m, meaning $\vec{\mathbf{D}}_{Net}$ = 2.00 m north.

This may be represented graphically by: (1) Drawing a point and labeling it "origin". (2) Drawing a line from the origin and labeling it "+". (3) Drawing displacement vector 1 with its tail at the origin and pointing in the positive direction (its tip should be on the positive axis). (4) Displacement vector 2 is drawn with its tail at the tip of vector 1 and pointing in the same direction. (5) The sum of these two vectors is a vector that has its tail at the same position as the tail of displacement vector 1 and its tip at the same position as the tip of vector 2. This is illustrated below.



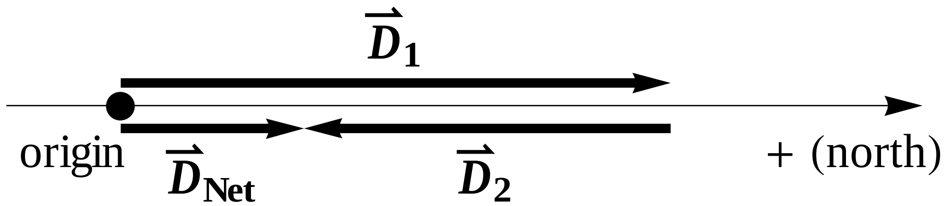
You will notice that the displacement vectors have been offset vertically. This is often done when drawing vector addition in one dimension. The three vectors are collinear and the net displacement vector is right on top of displacement vectors 1 and 2. However, drawing it right on top conceals where it begins and ends.

We have also worked with vector addition of vectors in opposite direction. Suppose after walking 1.20 m north, you walked 0.80 m south. If we keep north as our positive direction, then your net displacement would be

$$\vec{\mathbf{D}}_{Net} = \vec{\mathbf{D}}_1 + \vec{\mathbf{D}}_2 = +1.20 \text{ m} + (-0.80 \text{ m}) = +0.40 \text{ m} = 0.40 \text{ m north}$$

Note: We are still **adding** the vectors!

We can represent this graphically by following the same procedure outlined for two vectors in the same direction. The only change we make is to point displacement vector 2 in the negative direction.



Again, we have offset the vectors vertically for clarity.

Our procedure can be generalized for adding any number of vectors. You just always start the first at the origin and begin each subsequent one at the tip of the previous. The vector sum points from the origin to the tip of the last vector in the sum.

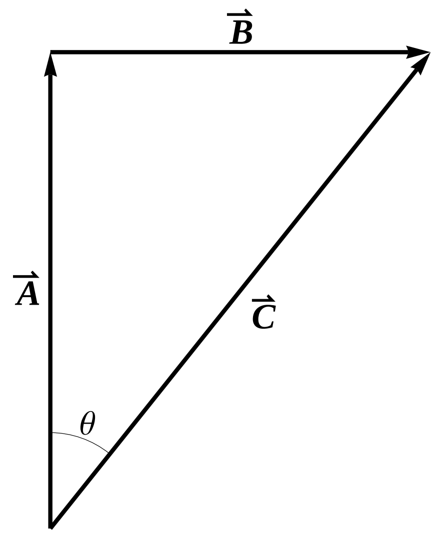
Exercise 1: A runner starts a race that consists of three legs. For the first leg, she runs 2 miles due north. The second leg consists of running 9 miles due south. For the final leg, she runs another 2 miles due north. What is her total displacement? Solve both algebraically and graphically.

Exercise 2: You are paddling a canoe down a river. Your canoe moves at a velocity of 2.5 m/s west relative to the water and the water in the river is moving at 0.8 m/s west relative to the bank. What is the velocity of your canoe relative to the bank? To answer this question, add the velocity of the water relative to the bank to the velocity of the canoe relative to the water.

Exercise 3: You stop paddling your canoe as you head down the river. Your boat now moves with the water at 0.8 m/s relative to the bank. You observe a pelican flying at 4.0 m/s east relative to your boat. What is the velocity of the pelican relative to the bank? Again, answer by adding the velocity of your canoe relative to the bank to the velocity of the bird relative to your canoe.

Two-dimensional vector addition: The graphical method of addition of two vectors is the same as for the one-dimensional case that is the first vector is represented by an arrow with a length proportional to the magnitude of the first vector and pointing in the correct direction. The second vector is placed with its tail at the tip of vector 1, pointing in the correct direction and with a length proportional to its magnitude. The tail of the resultant vector is placed at the location of the tail of the first vector. The tip of the resultant vector is placed at the location of the tip of the last vector. The length of the resultant vector will then be proportional to the magnitude of the resultant vector and it will be pointing in the correct direction.

As mentioned previously, the addition of two vectors that are perpendicular to each other is the "easiest" example of two-dimensional vector addition. We will use the trigonometric relationships and the Pythagorean Theorem to determine the magnitude and direction of the resultant vector.



The diagram above shows the vector $\vec{\mathbf{C}}$ as the **vector sum** of vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$. Vector $\vec{\mathbf{C}}$ is called the resultant vector, which simply means the sum. Written as an equation, $\vec{\mathbf{C}} = \vec{\mathbf{A}} + \vec{\mathbf{B}}$ Note: $\mathrm{C} \neq \mathrm{A} + \mathrm{B}$. That is, the magnitude or the resultant vector is not in general equal to the sum of the magnitudes of the two vectors. (What is the exception to this general rule?)

The correct relationship between the magnitude of the resultant vector and the magnitudes of the two vectors being added is the same as the relationship between a vector and its components:

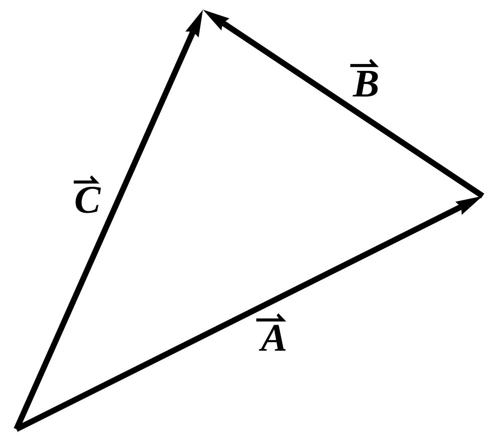
In the figure, $\mathrm{C}^2 = \mathrm{A}^2 + \mathrm{B}^2$ and $\theta = \tan^{-1}\left(\frac{\mathrm{B}}{\mathrm{A}}\right)$.

We have seen this interpretation already when we discussed vector components.

Exercise 4: A family on vacation in San Francisco drives from Golden Gate Park due south on 19th Avenue for 2.2 miles and then turns west on Sloat Boulevard and drives an additional 1.1 mile to go to the zoo. The driving time for this trip is 18 minutes. What is the family's net displacement for this trip? What is the average speed for the trip? What is the average velocity? (Remember to specify the direction of the net displacement and the velocity vectors. This requires giving a reference, like "of north", and a direction the angle is measured, like "east" in the direction "20.0° east of north".)

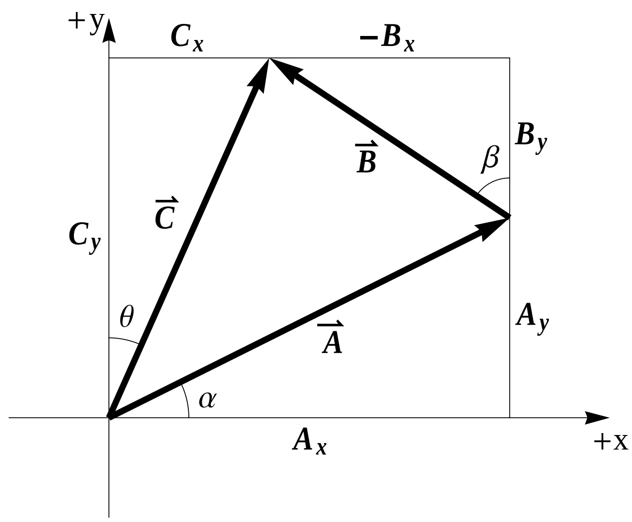
Exercise 5: A pelican flies 12 meters due north and then dives straight downward 24 meters in order to catch a fish. This entire maneuver takes 5.0 seconds. What is the net displacement of the pelican from its starting position? What is its average speed during this time? What is its average velocity during this time?

It is also possible to add vectors that are not at right angles to each other. The graphical method is the same as in all previous cases. The first vector is represented by an arrow with a length proportional to its magnitude and pointing in the correct direction. The second vector is placed with its tail touching the tip of vector 1, pointing in the correct direction and with a length proportional to its magnitude. The tail of the resultant vector is placed at the location of the tail of the first vector. The tip of the resultant vector is placed at the location of the tip of the last vector. The length of the arrow representing the resultant vector will then be proportional to the magnitude of the resultant vector and will be pointing in the correct direction.



The above figure depicts the vector equation $\vec{\mathbf{C}} = \vec{\mathbf{A}} + \vec{\mathbf{B}}$.

To add these vectors algebraically, we must first break them into components in an appropriate rectangular coordinate system. Then the components of the resultant vector will be the sums of the components of the vectors being added. This will be represented graphically and algebraically below.



From the above diagram, we can see that $\mathrm{C_x = A_x + B_x}$ and $\mathrm{C_y = A_y + B_y}$ where $\mathrm{A_x = A\cos\alpha, B_x = -B\sin\beta, A_y = A\sin\alpha,}$ and $\mathrm{B_y = B\cos\beta}$. The magnitude and direction of the resultant vector $\vec{\mathbf{C}}$ can be found from its components using: $\mathrm{C^2 = C_x^2 + C_y^2}$ and $\theta = \tan^{-1}\left(\frac{\mathrm{C_x}}{\mathrm{C_y}}\right)$.

Notice the angle θ found is the direction of $\vec{\mathbf{C}}$ from the +y-direction toward the +x-direction. Also, notice that $\mathrm{B_x = -B\sin\beta}$. This came from the definition of the sine of the angle β, $\sin\beta = \frac{|\mathrm{B_x}|}{\mathrm{B}}$, since the x-component of B is negative, its absolute value is -$\mathrm{B_x}$. $\sin\beta = \frac{-\mathrm{B_x}}{\mathrm{B}}$ means $\mathrm{B_x = -B\sin\beta}$.

Exercise 6: A treasure map gives the following directions:

(1) Walk fifty meters at 30^o north or east from the old oak tree.
(2) Turn 45^o to your left (you should now be facing 75^o north of east) and walk another fifty meters. You will find the treasure buried under a rock.

What straight-line path would take you directly from the old oak tree to the rock with the treasure buried under it?

Exercise 7: You followed the first direction on the treasure map correctly but after trying to follow the second direction, you do not find a rock with treasure buried under it. Instead, you are standing in a mudhole 92.4 meters 7.50^o north of east from the oak tree. What was the erroneous path (distance and direction) you took for the second part of the directions? What mistake did you make in following the directions on the treasure map?

Exercise 8: A cross-country skier is skiing across a level snow-covered field. She first skies half a mile west, turns and skies one mile northwest (45^o north of west), and then skies and additional two miles northeast (45^o north of east). What is her net displacement (direction and magnitude) from her starting position? What average velocity would she need to have to return (in a straight line) to her starting position in fifteen minutes?