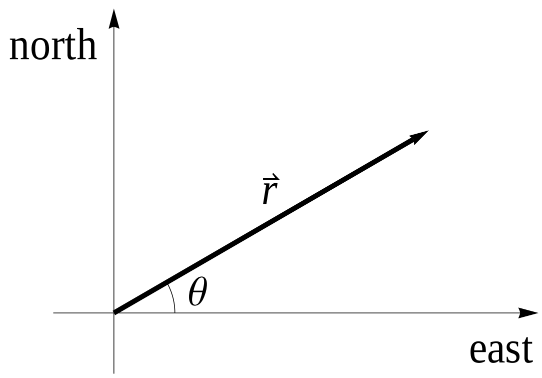
Vectors and Vector Components

Many physical quantities in physics are represented by vectors. A vector quantity has both a magnitude (numerical value, including units) and a direction. Examples of vector quantities include displacement, velocity, acceleration, and force. Frequently, vectors are represented graphically by arrows. The length of the arrow corresponds to the magnitude of the vector while the direction of the arrow represents the direction of the vector.

It is often useful to find the components of a vector in a Cartesian coordinate system. This requires the use of trigonometry to relate the magnitudes of the components of a vector to the magnitude and direction of the vector. Here is a typical example.

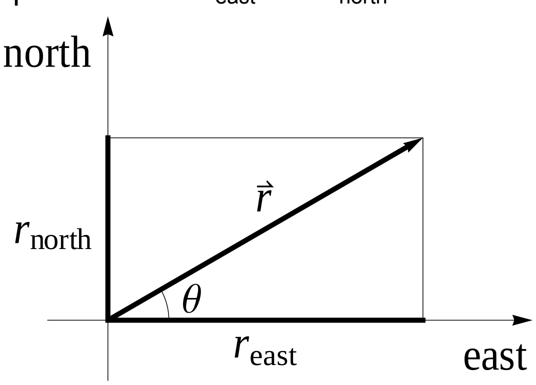
Example 1: A person walks 300 meters along a straight path directed 30° north of east. How far east and how far north is she from her starting position?

In this case, the vector quantity is a displacement. The coordinates being used are east and north. The displacement vector, $\vec{\mathbf{r}}$, may be represented as shown below:

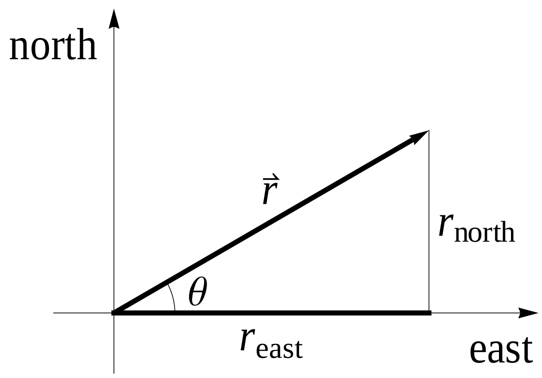


The magnitude of $\vec{\mathbf{r}}$**,** represented by $|\vec{\mathbf{r}}|$ or r, is 300 meters. The direction is given by the angle θ which is 30° north of east.

To find the components of $\vec{\mathbf{r}}$, draw lines from the head of **r** perpendicular to the east and north axes. Then, draw in the components of **r:** $\mathbf{r}_{east}$ and $\mathbf{r}_{north}$.



Note: The vector $\vec{\mathbf{r}}$ is the diagonal of a rectangle, so the side opposite the angle θ in the drawing has the same length as $\mathbf{r}_{north}$. We can now obtain the lengths of $\mathbf{r}_{east}$ and $\mathbf{r}_{north}$ from the magnitude of $\vec{\mathbf{r}}$ and the angle θ using trigonometric definitions.



Using the definitions of the trigonometric functions gives the following relationships between the magnitude of vector **r** and the magnitudes of its components **r**$_{\text{east}}$ and **r**$_{\text{north}}$.

$$\cos\theta = \frac{r_{east}}{r} \qquad \text{or} \qquad r_{east} = r\cos\theta$$

and

$$\sin\theta = \frac{r_{north}}{r} \qquad \text{or} \qquad r_{north} = r\sin\theta\,.$$

In this case: $r_{east} = r\cos\theta = (300m)\cos 30° = 260m$

and

$r_{north} = r\sin\theta = (300m)\sin 30° = 150m\,.$

She is 260 meters east and 150 meters north of her starting position.

In general, it is possible for **r**$_{\text{east}}$ to be positive or negative. It will be positive if $\vec{\mathbf{r}}$ is in the 1st or 4th quadrant and negative if $\vec{\mathbf{r}}$ is in the 2nd or 3rd quadrant. Similarly, **r**$_{\text{north}}$ will be positive if $\vec{\mathbf{r}}$ is in the first or second quadrant and negative if $\vec{\mathbf{r}}$ is in the 3rd or 4th quadrant

Now, if the components of a vector are given and we wish to find its magnitude and direction, we would use the Pythagorean Theorem and an inverse tangent function.

$$r^2 = r_{east}^{\ \ 2} + r_{north}^{\ \ 2} \qquad \text{or} \qquad r = \left(r_{east}^{\ \ 2} + r_{north}^{\ \ 2}\right)^{\frac{1}{2}}$$

and

$$\tan\theta = r_{north}/r_{east} \qquad \text{or} \qquad \theta = \tan^{-1}\left(\frac{r_{north}}{r_{east}}\right)^{*}.$$

In this case: $r = \left(r_{east}^{\ \ 2} + r_{north}^{\ \ 2}\right)^{\frac{1}{2}} = \left[(260m)^2 + (150m)^2\right]^{\frac{1}{2}} = 300m$

and

$\theta = \tan^{-1}\left(r_{north}/r_{east}\right) = \tan^{-1}\left(150m/260m\right) = 30°\,.$

*One must be careful when using a calculator to compute an inverse trigonometric function. The inverse sine and tangent functions on a calculator always report an angle with in the 1st or 4th quadrant. Try this: (1) Use your calculator to take the tangent of 225°. (2) Now take the inverse tangent of your result. Your calculator should say 45°. The reason for this is there are actually an infinite number of solutions for θ to the equation $\tan\theta = 1$. In degrees, they are $\theta = 45° + n(180°)$, where n is any integer. Try tan(-135°), tan(405°), and as many more as you want. The important thing to do is ***always*** draw a picture which will show you which quadrant the vector is in, then make sure when you interpret from your calculator: $\theta = \tan^{-1}(1) = 45°$ means $\theta = 45° + n(180°)$.

Exercise 1: A block is pulled a distance of 2.2 meters up a ramp inclined at an angle of 50° above horizontal. What are its horizontal and vertical displacements from its starting position?

Exercise 2: A boat travels 25 miles down a river that runs 60° south of west. What are the components of the boats displacement?

Exercise 3: A sled is pulled along a horizontal snow-covered path by a rope that makes an angle a 45° with the horizontal. The rope exerts a force of 10 pounds on the sled. Find the horizontal and vertical components of the force the rope exerts on the sled.

Exercise 4: Two forces are applied to an object in directions perpendicular to each other. One of the forces has a magnitude of 20 N (newtons) and the other has a magnitude of 15 N. These two force vectors can be considered the components of the net force vector. What is the net force (magnitude and direction) acting on the object?

Exercise 4 is solved by considering the two forces as the components of the net force vector. Alternatively, the net force vector may be considered to be the sum of the two individual force vectors. This is one of the "easiest" examples of the addition of two vectors. Even "easier" examples are the addition of vectors that lie along the same direction.

We will explore vector addition in the next worksheet.