Systems of Equations

<u>Multiple Equations with Multiple Unknowns</u>: The general rule that you need to be aware of is that to solve for two unknowns, you need two independent equations containing those two unknowns (and no other unknowns). More generally, you need n independent equations to solve for n unknowns (where n is any number—any positive integer, to be precise). The techniques for solving for the unknowns are illustrated below.

<u>Independent Equations</u>: If you can obtain one (or more) of your equations by algebraically combining the other equations, the set is *not* independent. This is not always easy to see, but it is important to be aware of. It will certainly come up in your physics courses.

Example 1: Is this set of equations independent or not? That is, can you get one of these equations as a combination of the other two equations?

(1) $A + B = C$ (2) $A - B = D$ (3) $2A = C + D$

This set is NOT independent. If you add equation (1) and equation (2), the result is equation (3). Thus, equation (3) provides no information beyond what is given by equations (1) and (2) and you cannot solve for three unknown variables using this set of equations. In fact, you should show that combining any two equations yields the third. Any two of these equations do make an independent set, so you could solve for two unknowns.

Exercise 1: For each set of given equations, how many independent equations are there? How many unknowns could you solve for?

(a) (1) $A + B = C$ (2) $A - B = D$ (3) $C + 3D = 7$

(b) (1) $V_1 - I_1 R_1 - I_2 R_2 = 0$ (2) $V_1 - I_1 R_1 - I_3 R_3 = 0$ (3) $I_3 R_3 - I_2 R_2 = 0$

<u>Solving Multiple Equations</u>: The general plan is to algebraically combine the equations so that you end up with one equation with only one unknown. You then solve for that unknown and use it to work back through your other equations to find the other unknowns. The two general methods for doing this are **substitution** and **elimination.**

General Procedure for Substitution Method

1. Solve one equation for one unknown in terms of the other(s).
2. Then substitute the result into another equation and solve it for the next unknown.
3. Continue until you have an expression for one unknown in terms of known quantities.
4. Then work backwards to find the other unknowns.

General Procedure for Elimination Method

1. Add one equation or a multiple of one equation to a second equation to eliminate one of the unknowns.
2. Repeat this process until you have one equation in one unknown and solve for that unknown.
3. It is also sometimes useful to divide one equation by another to eliminate one of the unknowns.

Let's look at a few examples that illustrate the methods that you will be using. We will first solve a system of equations by substitution to illustrate that method. For simplicity, we will first work with two independent equations and two unknowns.

Example 2: Solve the following system of equations for the unknowns B and C:

$$(1)\ \ A + B - C = 2D \qquad (2)\ \ B + 2C = D$$

Note that A and D are considered known quantities.

Step 1: Solve for one of the unknowns in terms of the other from either equation.
With practice, you learn to spot what might be the easiest approach.
In this case, we can solve for either B or C from either equation.
For this example, let's use equation (2) to get an expression for B.

Subtract $2C$ from each side of equation (2) and obtain: $B = D - 2C$

Step 2: Replace B in equation (1) with $D - 2C$. The result will be an equation with only C as an unknown.

$A + B - C = 2D$ becomes $A + (D - 2C) - C = 2D$ which is

$$A - 3C = D$$

Now isolate the unknown, C: $\quad A - D = 3C$

And solve! $\quad C = (A - D)/3$

Since A and D are known, now C is now known.

Step 3: Let's find B by substituting this expression for C back into ***either*** of the original equations ***or*** the equation we found at the end of Step 1.

At the end of Step 1, we found $B = D - 2C$. Substituting the expression in for C we obtain

$$B = D - 2[(A - D)/3] = D - \frac{2}{3}A + \frac{2}{3}D, \text{ so } B = \frac{5}{3}D - \frac{2}{3}A$$

(Notice that the -2 when distributed gave us a +2/3 D.)

We can now report the solution as $B = \frac{5}{3}D - \frac{2}{3}A$ and $C = \frac{A - D}{3}$.

The system of equations has now been solved. We were told that we already knew A and D, so we found expressions for B and C in terms of them. Any correct algebraic manipulations of the equations would have gotten us to the exact same results. But don't take my word for it, check it out yourself!

Exercise 2: For the same two equations, (1) $A + B - C = 2D$ and (2) $B + 2C = D$, solve for B and C using the following steps.

Step 1: From equation (1), find an expression for C.
Step 2: Substitute the expression for C found in step one in equation (2).
Step 3: Solve for B using the new form of equation (2).
Step 4: Substitute the expression for B found in step 3 into either original equation and solve for C.
Step 5: Compare your results to those obtained in the example. If they are not the same, one of us made a mistake.

Exercise 3: For the same two equations, solve for B and C by using the following steps.

Step 1: From equation (2), find an expression for C.
Step 2: Substitute the expression found in step one for C in equation (1).
Step 3: Solve for B using the new form of equation (1).
Step 4: Substitute the expression for B found in step 3 into either original equation and solve for C.
Step 5: Compare your results to those obtained previously.

Example 2 and Exercises 3 and 4 should illustrate for you that no matter which approach you choose, all the substitution methods yield the same solution.

Now that you've seen and practiced substitution, let's look at the elimination method. We will continue to work with the same system of equations and the same unknowns.

Example 3: Solve the following system of equations for the unknowns B and C:

$$(1)\ \ A + B - C = 2D \qquad (2)\ \ B + 2C = D$$

Step 1: To get rid of B, subtract equation (2) from equation (1). This is the same as multiplying equation (2) by negative 1and then adding the equations.

$$\begin{array}{cr} (1) & A + B - C = 2D \\ -1\times(2) & -B - 2C = -D \\ \mathit{Sum} & A - 3C = D \end{array}$$

Now we have an equation which does not contain the unknown B. We have **eliminated** B.

Step 2: Solve for C. It is easy to see that we obtain the same result as we did before.

$$C = (A - D)/3$$

Step 3: As before, substitute this expression into either equation and find B. The result, of course, will be identical to that obtained in all of the previous solutions.

Okay, your turn.

Exercise 4: Solve the same system of equations for B and C.

Step 1: Add a multiple of equation (1) to equation (2) to eliminate C.
Step 2: Solve the resulting equation for B.
Step 3: Substitute your expression for B back into equation (1) or (2) and solve for C.

The equations we encounter in physics are not always linear. The next example illustrates both methods applied when one equation contains a quadratic term with an unknown.

Example 4: Solve $x = x_0 + v_0 t + ½ a t^2$ and $v = v_0 + a t$ for v_0 and t.

> *Substitution: We could solve either equation for the unknown* v_0 *or the unknown* t*. In general, we will want to choose to solve the simpler equation (in this case not the quadratic!) for one unknown. If we solve the second equation for* t*, we will have to substitute it into two places in the first equation* ***and*** *we would have to square it when substituting into the* $½ a t^2$ *term. If we solve instead for* v_0*, we will only have to substitute it into one place in the first equation (and not square it!). Here we go!*

1. Solve the second equation for v_0. $v_0 = v - a\,t$
   This can be done in 1 algebraic step! What algebraic step was required?
2. Substitute the result into the first equation. $x = x_0 + (v - a\,t)\,t + \frac{1}{2}a\,t^2$
   We now have an equation with only one unknown!
3. Solve for t:

$$x = x_0 + v\,t - a\,t^2 + \tfrac{1}{2}a\,t^2 = x_0 + v\,t - \tfrac{1}{2}a\,t^2$$

$$x - x_0 - v\,t + \tfrac{1}{2}a\,t^2 = 0$$

$$t = \frac{-(-v) \pm \sqrt{v^2 - 4\left(\frac{1}{2}a\right)(x - x_0)}}{2\left(\frac{1}{2}a\right)} = \frac{v \pm \sqrt{v^2 - 2a(x - x_0)}}{a}$$

4. Substitute the result for t back into one of the original equations or the equation from step 1.

$$v_0 = v - a\left(\frac{v \pm \sqrt{v^2 - 2a(x - x_0)}}{a}\right) = v - \left(v \pm \sqrt{v^2 - 2a(x - x_0)}\right) = \mp\sqrt{v^2 - 2a(x - x_0)}$$

The solution can now be reported as $t = \dfrac{v + \sqrt{v^2 - 2a(x - x_0)}}{a}$ and $v_0 = -\sqrt{v^2 - 2a(x - x_0)}$ or

$t = \dfrac{v - \sqrt{v^2 - 2a(x - x_0)}}{a}$ and $v_0 = +\sqrt{v^2 - 2a(x - x_0)}$.

Note, we have to correctly pair up our variables $t = \dfrac{v + \sqrt{v^2 - 2a(x - x_0)}}{a}$ and

$v_0 = +\sqrt{v^2 - 2a(x - x_0)}$ is **NOT** a solution.

Elimination:

1. Multiply the second equation by -t. $-v\,t = -v_0\,t - a\,t^2$
2. Add the result to the first equation.

$$\begin{aligned} x &= x_0 + v_0\,t + \tfrac{1}{2}a\,t^2 \\ -v\,t &= -v_0\,t - a\,t^2 \end{aligned}$$

Sum: $x - v\,t = x_0 - \frac{1}{2}a\,t^2$

Notice that the v_0 t and the $-v_0$ t cancelled.

Again, we have an equation with only one unknown!

3. Solve for t:

$$x - v\,t = x_0 - \tfrac{1}{2}a\,t^2$$

$$x - x_0 - v\,t + \tfrac{1}{2}a\,t^2 = 0$$

This is the exact same equation we applied the quadratic formula to solve in the previous method. From here the steps are the same as the substitution method. After you have t, substitute back into one of the original equations and solve for v_0

Now try your hand at these. Pick whichever method you like. If you're feeling ambitious, try a few different approaches for each. Note that those variables *not* identified as unknown are to be treated as known.

Exercise 5: Solve the system of equations (1) $\lambda + 2\mu = 11\omega$ and (2) $4\mu + \omega = 7\lambda$ for λ and μ. (Which variable is considered known?)

Exercise 6: Solve the system of equations (1) $y = 2x + 2$ and (2) $y = 8x - 1$ for x and y.

Once you have solved for x and y algebraically, carefully sketch the curves for each equation on the same graph. Find the point at which the curves intersect and compare with your algebraic result.

OK, the next exercise is here to illustrate a special case that sometimes comes up when solving systems of equations.

Exercise 7: Solve the system of equations (1) $2x - y = 0$ and (2) $y = 2x + 2$ for x and y.

First, see what happens when you solve algebraically by substitution or elimination. What does this mean? Can these two equations both be true at the same time?

After you have considered the problem algebraically, graph each equation and see if your algebraic findings make more sense to you.

So far the exercises you've been asked to do have turned out fairly simple. Real world problems are not always simple.

Exercise 8: Solve the system of equations (1) $2p^2t - 4r = 7$ and (2) $2t - 1 = r + 3$ for r and t.

Use substitution or elimination to solve.

Here is what you should find: $r = \dfrac{7 - 4p^2}{p^2 - 4}$ and $t = \dfrac{-9}{2p^2 - 8}$.

Now, this is not the only way the solutions can be written, if you get something different, check that you get these expressions if you write r and t as a single fraction.