Review of Exponential and Logarithmic Functions

Definition: For $b > 0$ and $b \neq 1$, and all real numbers x,

$$y = b^x$$

is an exponential function with "base b."

Property: For $b > 0$ and $b \neq 1$, if $b^x = b^y$, then $x = y$.

Definition: For $b > 0$ and $b \neq 1$, and all positive numbers x,

$$y = \log_b x$$

means the same as $x = b^y$.

Properties: If $b > 0$ and $b \neq 1$, then $\log_b b^x = x$.

If $b > 0$ and $b \neq 1$ and $x > 0$, then $b^{\log_b x} = x$.

For $b > 0$ and $b \neq 1$, $\log_b b = 1$ and $\log_b 1 = 0$.

Rules of Logarithms

Product Rule: If x, y, and b are positive numbers, $b \neq 1$, then

$$\log_b xy = \log_b x + \log_b y.$$

Quotient Rule: If x, y, and b are positive numbers, $b \neq 1$, then

$$\log_b (x/y) = \log_b x - \log_b y.$$

Power Rule: If x and b are positive real numbers, $b \neq 1$, and if n is any real number, then

$$\log_b x^n = n \log_b x.$$

Verifying the rules of logarithms.

To prove the product rule, rewrite the logarithmic statements in exponential form

$\log_b x = m$ means $b^m = x$ and $\log_b y = n$ means $b^n = y$.

So, by substitution, and using the product rule for exponents,

$$xy = b^m \cdot b^n = b^{m+n}.$$

Rewrite this in logarithmic form,

$$\log_b xy = m + n.$$

Now substitute for m and n to get

$$\log_b xy = \log_b x + \log_b y.$$

These rules and properties can be used to solve algebra problems as shown in the following example.

Example: Solve the following equation for x: $5^{x+3} = 25$.

Take $\log_5$ of both sides of the equation: $\log_5 5^{x+3} = \log_5 25$

This leads to $x + 3 = \log_5 5^2$ which becomes $x + 3 = 2$. Solving gives $x = -1$.

Commonly Confused Expressions

(1) $\log_b (x \pm y)$ is not the same as $\log_b x \pm \log_b y$.

(2) $\log_b (xy)$ is not the same as $(\log_b x)(\log_b y)$.

EXERCISES

1. a. For each statement given in exponential form, rewrite it in logarithmic form.

   i. $2^x = 4$ ii. $10^5 = 100{,}000$ iii. $4^{-3} = 1/64$

   b. For each statement given in logarithmic form, rewrite it in exponential form.

   i. $\log_2 8 = 3$ ii. $\log_{10} 1{,}000{,}000 = 6$ iii. $\log_3 \frac{1}{9} = -2$.

2. Solve the following equations for the given variable.

   a. $2^x = 4$ b. $9^x = 27$ c. $4^x = 64$ d. $3^x = \frac{1}{9}$

   e. $\left(\frac{1}{2}\right)^x = 8$ f. $6^{-x} = \frac{1}{6}$ g. $2^{3x+2} = 16$ h. $16^{(x+1)/2} = 32$

Problems 3 and 4 may be done as in-class exercises. Check with your instructor.

3. Show why for the first property (For $b > 0$ and $b \neq 1$, if $b^x = b^y$, then $x = y$.), the statement "$b \neq 1$" is necessary.

4. Verify the quotient rule and the power rule.

5. Evaluate the following.

   a. $\log_{10} 1{,}000$ b. $\log_{10} 0.001$ c. $\log_{10} 1$ d. $\log_8 64$

   e. $\log_5 125$ f. $\log_3 (1/27)$ g. $\log_5 25^2$ h. $\log_3 27^{1/2}$

6. Solve each equation for the given variable.

   a. $y = \log_6 216$. b. $\log_5 x = -3$ c. $\log_x 9 = 1/2$ d. $\log_c 125 = -3$

   e. $\log_4 x = 5/2$ f. $\log_2 \mu = 0$ g. $\log_p 0.1 = 1$ h. $\log_{1/2} f = 1$

7. Write each of the following as a single logarithm. Assume all variables represent positive real numbers. Simplify as much as possible.

   a. $\log_6 2 + \log_6 3$ b. $\log_{10} 50 - \log_{10} 5$ c. $3\log_5 2 + \log_5 6^2$

8. Find examples that illustrate the commonly confused expressions.