Algebraic Equations

Types of Equations and Solutions

Solutions to Equations are values for the variable or variables that make the equation true.

Examples: x = 3 is a solution to the equation x + 2 = 5, because 3 + 2 = 5

x = 3 and y = 2 is a solution to the equation y = x – 1, because 2 = 3 – 1

x = 3 is a solution to the equation $x^2 = 9$, because $3^2 = 9$

x = -3 is also a solution to the equation $x^2 = 9$, because $(-3)^2 = 9$

Note: When we list multiple solutions to a single equation, we separate the solutions with the word or. Since $x^2 = 9$ is true if x = 3 or x = -3, we say that the equation $x^2 = 9$ is solved if x = 3 or x = -3. In this case we can save writing with the plus-or-minus-symbol, "±", and write the solutions to $x^2 = 9$ as x = ±3.

Not all equations have solutions. In this class, we will restrict ourselves to real solutions to equations. For example, $x^2 = -1$ has no real solutions. An equation with no solutions is called a contradiction. An example of a contradiction is the equation x = x + 2. There is no value you can substitute for x and have both sides of the equation be the same value.

Some equations are true for some, but not all, values of the variable or variables. Such an equation is called a conditional. An example of a conditional is x = x + y. This equation is true provided y = 0, but is not true if y is any other value.

Other equations are true for all values of the variable or variables. Such an equation is called an identity. With an identity, one side of the equation can be rearranged into the other side of the equation. Examples include distribution, 2(x - 3) = 2x - 6, obtaining a common denominator, $\frac{x}{2} + \frac{1}{6} = \frac{3x+1}{6}$, definitions, $a^3 = a \cdot a \cdot a$, and rules, $a^3 \cdot a^m = a^{3+m}$.

A. Classify each of the following equations as an identity, a conditional or a contradiction. Assume all variables are positive real numbers.

1. $\frac{A}{B} + \frac{C}{D} = \frac{AD + CB}{BD}$

2. $\frac{A}{C} + \frac{B}{D} = \frac{A + B}{C + D}$

3. $\frac{\lambda}{\alpha + \beta} = \frac{\lambda}{\alpha} + \frac{\lambda}{\beta}$

4. $\frac{\alpha + \beta}{\lambda} = \frac{\alpha}{\lambda} + \frac{\beta}{\lambda}$

5. $\frac{(G+Q)/(k+d)}{(k+d)/(G+Q)} = 1$

6. $\frac{\sigma}{\theta/\phi} = \frac{\sigma \cdot \phi}{\theta}$

7. $\frac{\sigma}{\theta/\phi} = \frac{\sigma \cdot \theta}{\phi}$

8. $\frac{R}{\sigma} \times \frac{S + T}{\beta} = \frac{RS + RT}{\sigma\beta}$

9. $\frac{R}{\sigma} \div \frac{S + T}{\beta} = \frac{R\beta}{\sigma S + T}$

10. $\frac{1}{C_1} + \frac{1}{C_2} = \frac{C_1 C_2}{C_1 + C_2}$

Solving an Equation for an Unknown Variable

Two equations with the same solutions are said to be equivalent. An example of a pair of equivalent equations is x + 2 = 5 and x = 3. Both of these equations are true only if x = 3.

Given any equation (x = 3), you can obtain an equivalent equation by doing any of the following:

Add the same thing to both sides of the equation, x + 2 = 3 + 2 or x + 2 = 5
Subtract the same thing from both sides of the equation, x - 1 = 3 - 1 or x - 1 = 2
Multiply both sides of the equation by the same thing (except 0!), 2·x = 2·3 or 2x = 6
Divide both sides of the equation by the same thing (except 0!), x÷3 = 3÷3 or $\frac{1}{3}$x = 1

Of course subtraction is the same as addition of a negative and division is the same as multiplication by a reciprocal, so we don't really need all of the examples, as long as we remember not to multiply or divide by zero.

The equations x = 3, x + 2 = 5, x - 1 = 2, 2x = 6, and $\frac{1}{3}$x = 1 are all equivalent, because they are true only for x = 3. That is to say, x = 3 is the only solution to each of these equations.

If we are asked to solve the equation x + 2 = 5 for x, we would subtract 2 from both sides and have x + 2 - 2 = 5 - 2 which simplifies to x = 3.

To solve 2x = 6 for x, we could multiply both sides by ½. (½)(2x) = (½)(6) which simplifies to x = 3. Instead of multiplying by ½, we could have divided by 2 and obtained the same result.

Sometimes to solve, we must perform more than one step. Suppose we wish to solve the equation $2a+b=12$ for a. This equation contains two variables, a and b. When we solve for a, we are considering b to be a known quantity. We may have a single value for b or we may have a list of values, i.e. "Determine the value of a when b is 1, 2, 3, 4, and 5". (We will see in a later section that this can be useful when graphing the solutions for an equation with two variables, where we will solve an equation for one variable and plug in values for the other variable to determine ordered pairs to plot.)

$$\begin{aligned} 2a+b&=12 \\ 2a+b-b&=12-b \\ 2a&=12-b \\ \tfrac{1}{2}\cdot 2a&=\tfrac{1}{2}\cdot(12-b) \\ a&=6-\tfrac{b}{2} \end{aligned}$$

Not all solutions can be found with these simple steps. Sometimes we must use exponents or roots, the quadratic formula, or other methods to find a solution.

Example Requiring a Root: The solutions to the equation $x^2 = 9$ are found by taking the square root of each side of the equation. When we use a square root, we have to remember that we will have both a positive and a negative solution, x = ±3 in this case.

Example Requiring an Exponent: The solution to the equation $\sqrt{b}=9$ is found by squaring both sides of the equation. The result in this case is $b=81$.

Quadratic formula: The quadratic formula is frequently used to solve algebraic problems in physics.

When the quadratic formula is written as $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, it is the solution to the quadratic equation $ax^2 + bx + c = 0$ for the variable x. The equation $ax^2 + bx + c = 0$ contains 4 variables, but only x is considered unknown.

Simple Examples: Solve $2x^2 - 5x + 2 = 0$ for x.

$$x = \frac{-(-5) \pm \sqrt{(-5)^2 - 4 \cdot 2 \cdot 2}}{2 \cdot 2} = \frac{5 \pm \sqrt{9}}{4} = \frac{5 \pm 3}{4} = \frac{8}{4} \text{ or } \frac{2}{4} = 2 \text{ or } \frac{1}{2}$$

Solve $4h^2 - 12h = 0$ for h.

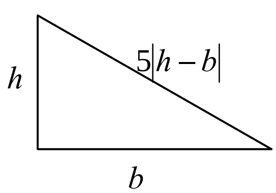
$$h = \frac{-(-12) \pm \sqrt{(-12)^2 - 4 \cdot 4 \cdot 0}}{2 \cdot 4} = \frac{12 \pm 12}{8} = \frac{24}{8} \text{ or } \frac{0}{8} = 3 \text{ or } 0$$

Notice that in the second example, "c" was zero in the quadratic formula because there was no constant term in the quadratic equation $4h^2 - 12h = 0$. When using the quadratic formula, "b" and/or "c" can be zero but "a" cannot. Why?

The coefficients "a", "b" and "c" are not always given in the original equation and can contain variables.

More Complex Example requiring the Quadratic Formula:

A right triangle has a base b, a height h and a hypotenuse that is 5 times the difference between the height and base. Determine the height of the triangle. We will learn in a later section the steps for setting up geometric word problems. In this case we will draw a picture and apply the Pythagorean Theorem.



$$b^2 + h^2 = \left(5|h - b|\right)^2$$
$$b^2 + h^2 = 25h^2 - 50hb + 25b^2$$
$$0 = 24h^2 - 50bh + 24b^2$$

We now apply the quadratic formula with 24 as "a", (-50b) as "b" and 24b^2 as "c".

$$h = \frac{-(-50b) \pm \sqrt{(-50b)^2 - 4 \cdot 24 \cdot 24b^2}}{2 \cdot 24} = \frac{50b \pm \sqrt{2^2 25^2 b^2 - 2^2 24^2 b^2}}{2 \cdot 24}$$
$$= \frac{2 \cdot 25b \pm \sqrt{2^2 b^2 \left(25^2 - 24^2\right)}}{2 \cdot 24} = \frac{2 \cdot 25b \pm 2b\sqrt{(25 - 24)(25 + 24)}}{2 \cdot 24}$$
$$= \frac{2b}{2 \cdot 24} \cdot \left(25 \pm \sqrt{(1)(49)}\right) = \frac{b}{24}(25 \pm 7) = \frac{b}{24}(32) \text{ or } \frac{b}{24}(18) = \frac{4}{3}b \text{ or } \frac{3}{4}b$$

So the height is $\dfrac{3}{4}b$ or $\dfrac{4}{3}b$.

B. Solve for the specified unknown variable in each of the following examples:

1. Consider F to be unknown: $A + B - F = G + 2t$

2. Consider t to be unknown: $A + B - F = G + 2t$

3. Consider a to be unknown: $\dfrac{\theta + r}{a} - 2a = 0$

4. Consider λ to be unknown: $\dfrac{\alpha + \beta}{\lambda} = \dfrac{\lambda}{\alpha + \beta}$

5. Consider α to be unknown: $\dfrac{2\alpha + \beta}{\lambda} = \dfrac{\alpha - \beta}{\lambda}$

6. Consider d to be unknown: $\dfrac{b}{c + d} - 4cH = 3c$

7. More Challenging: Redo #6, considering c to be unknown.

Actual Physics Examples: Here are some actual physics equations that you will likely run into. Notice that you can do the algebra without knowing what the equations mean! (You *will* need to understand the physics to *set up* the equations so that they are ready to solve.)

Notice that, in some cases, you do *not* have numbers to "plug in," while in others you do. In the latter case, however, it is usually advisable to rearrange the symbols before putting in the numerical values. This serves at least two purposes:

1. It makes it easier to find mistakes. If you put in the numbers early on, erroneous algebraic steps are not easy to find.

2. You will end up with a *general* solution. A general solution can be used to investigate questions like "What if the object started with twice the speed?", "What angle gives the maximum distance?", "What if there were no friction?", and so on. A general solution is also useful in lab situations.

Be ready to use the two algebraic quick steps, to group terms and factor, to raise both sides to a power or take roots of both sides, and to use the quadratic equation.

C. Solve for the specified unknown variable in the following physics equations.
Assume that all other variables are known. If numerical values are given put them in at the end to obtain a numerical result. [Note: In these cases the known variables have a numerical value and units. Simplify the units as you would variables using the rules of exponents and roots.]

1. t = ?: v_1 = 30 m/s ,v_2 = 10 m/s, a = –10 m/s^2 : $v_2 = v_1 + at$

2. T_1 = ?: $Q = c\,m\,(T_2 - T_1)$

3. r = ? : $q\,v\,B = m\dfrac{v^2}{r}$

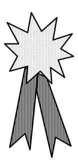
4. L = ?, g = 9.8 m/s^2, T = 1.0 s : $T = 2\pi\sqrt{\dfrac{L}{g}}$

5. R_{eq} = ?, R_1 = 2 Ω, R_2 = 4 Ω: $\dfrac{1}{R_1} + \dfrac{1}{R_2} = \dfrac{1}{R_{eq}}$

6. m = ?, F = 20.0 N, T = 4.0 N, μ_k = 0.15, g = 9.8 m/s^2, a = 3.0 m/s^2: $F - \mu_k\,m\,g + 0.866\,T = m\,a$

7. r = ? : $\dfrac{1}{2}m\,v^2 - G\dfrac{M\,m}{r} = \dfrac{1}{2}k\,x^2$

8. t = ?, Δy = 5.0 m, v_1 = 20 m/s, g = 10 m/s^2: $\Delta y = v_1 t - \frac{1}{2}gt^2$



If you can work through these fluently (accurately and quickly), you are well on your way towards mathematical preparedness for physics. If not, you can get there. Just practice, practice, practice until you get the game.