# REVIEW OF EXPONENTS

Definition: If a is a real number and n is a natural number,

$$a^n = \underbrace{a \cdot a \cdot a \cdots}_{\text{n factors of a}}$$

where a is the base and n is the exponent.

Definition: For any natural number n and any nonzero real number a, the negative exponent is defined by

$$a^{-n} = \frac{1}{a^n}.$$

## RULES OF EXPONENTS

Product Rule: If a is any real number and m and n are natural numbers, then

$$a^m \cdot a^n = a^{m+n}.$$

Quotient Rule: If a is any nonzero real number and m and n are nonzero integers, then

$$\frac{a^m}{a^n} = a^{m-n}.$$

Zero Exponent: If a is any nonzero real number, then $a^0 = 1$.
Note: $0^0$ is undefined.

Power Rules: If a and b are real numbers and m and n are integers, then

$$(a^m)^n = a^{mn}$$

$$(a\,b)^m = a^m\,b^m$$

$$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m} \quad (b \neq 0).$$

Note: The power rule, $(a\,b)^m = a^m\,b^m$, applies when raising a number with units to a power. The volume of a cubical box that is 2 ft on each side is $(2\text{ ft})^3 = 2^3\text{ ft}^3 = 8\text{ ft}^3$.

## COMMONLY CONFUSED EXPRESSIONS

(1) $-a^n$ and $(-a)^n$ do not mean the same thing.
(2) $ab^n$ is not the same as $(ab)^n$.
(3) $(a + b)^n$ is not the same as $a^n + b^n$.

## VERIFYING THE RULES OF EXPONENTS

The definitions of exponents and negative exponents may be used to verify the rules of exponents.

Example: The product rule.

$$
\begin{aligned}
a^m \cdot a^n &= \left(\underbrace{a \cdot a \cdot a \cdots}_{\text{m factors of a}}\right) \cdot \left(\underbrace{a \cdot a \cdot a \cdots}_{\text{n factors of a}}\right) \\
&= \underbrace{a \cdot a \cdot a \cdots}_{\text{m + n factors of a}} \\
&= a^{m+n}
\end{aligned}
$$

Exercises A and B will be done in class. They should be included with the rest of the exercises from this worksheet.

EXERCISE A. Verify the quotient rule, the zero exponent rule, and the power rules using the definitions.

## CHECKING THE COMMONLY CONFUSED EXPRESSIONS

If two expressions can be shown by any one example to yield different results, then the two expressions do not mean the same thing.

Example: (1) $-a^n$ and $(-a)^n$ do not mean the same thing.
We can illustrate this by choosing a = 3 and n = 2.
Now, $-a^n = -3^2 = -3 \cdot 3 = -9$, while $(-a)^n = (-3)^2 = (-3) \cdot (-3) = 9$

Since this example shows that the two expressions yield different results, the two expressions do not mean the same thing.

Note: You may be able to find examples where the two expressions yield the same result, even if the two expressions do not mean the same thing. You only need to find one example where the expressions yield different results to show that the expressions do not mean the same thing.

If a = 3 and n = 3,
then $-a^n = -3^3 = -3 \cdot 3 \cdot 3 = -27$ and $(-a)^n = (-3)^3 = (-3) \cdot (3) \cdot (-3) = -27$

Although in this example both expressions yield the same result, the counterexample given above (a = 3 and n = 2) shows that the two expressions do not mean the same thing.

EXERCISE B. Find examples (numerical values for a, b and n) that illustrate commonly confused expressions (2) and (3).

EXERCISE C. USING THE RULES OF EXPONENTS.

Simplify each expression. Assume that all variables used as exponents represent integers and that all other variables represent nonzero real numbers. Write all answers with only positive exponents. Evaluate whenever possible.

1. a. $(-1)^1$ b. $(-1)^2$ c. $(-1)^3$ d. $(-1)^4$ e. $(-1)^5$ f. $(-1)^6$

g. Do you see a pattern in the signs here? Explain this result in words. Does this pattern only work for -1 or does it work for any negative number?

h. $(-1)^n$ = ______ if n is oddi. $(-1)^n$ = ______ if n is even

j. $(-22)^n = 22^n$ if n is ________ k. $(-22)^n = -22^n$ if n is ________

2. a. $x^3x^5x^2$ b. $(3x^4)(-4x^3)$ c. $(-5a^6b^4)(-2a^3b^5)$

3. a. $2^{-3}$ b. $(3p^3)^{-4}$ c. $-2z^{-2}$ d. $5(-2r^3)(3r^{-5})$

4. a. $\dfrac{4^7}{4^3}$ b. $\dfrac{y^3}{y^8}$ c. $\dfrac{t^{6n}}{t^{4n}}$ d. $\dfrac{(4b)^{12}}{(4b)^{12}}$

5. a. $\left(\left[(3x)^2+2x\right]^3+x\right)^0$ b. $\dfrac{(3f^2)^4(2f^5)^2}{(9f^3)^6}$

6. a. $\dfrac{(-3\omega^2)^3(2\omega^{-3})^4}{(\omega^5)^{-6}}$ b. $\dfrac{\lambda^6+\lambda^2+\lambda^{-4}}{\lambda^3}$

7. a. $\dfrac{(p+q)^4(p+q)^2}{(p+q)^6}$ b. $\dfrac{3m^4(2m^5n^{-2})^2}{(4m^3)^3(6n^{-7})}$

8. a. $\dfrac{-(2c^3d^{-4})^5(-3^2cd^3)^3}{(-6^5c^{-4}d^6)^4}$ b. $\dfrac{(-8v^{-6}w^3)^{-2}(-4^2v^7w^{-2})^{-5}}{(2^5v^4w^3)^4}$

# REVIEW OF ROOTS AND RADICALS

Definition: For any real number x and natural number n, the real number r is an nth root of x if $r^n = x$.

| | |
|---|---|
| $\sqrt[n]{x}$ | denotes the nth root of x. |
| x | is the radicand. |
| $\sqrt{\ }$ | is the radical sign or the radical. |
| n | is the index, or order, of the radical. |

Square roots: If x is a real number and x > 0,

| | |
|---|---|
| $\sqrt{x}$ | is the positive (prinicipal) square root of x. |
| $-\sqrt{x}$ | is the negative square root of x. |

If x is any real number, then $\sqrt{x^2} = |x|$.

Fractional exponents: The nth root of x can be denoted by $x^{1/n}$ or $\sqrt[n]{x}$.

Therefore, $x^{1/n} = \sqrt[n]{x}$.

For all positive integers n, $0^{1/n} = 0$.

If x is positive and n is an even positive integer, then $x^{1/n}$ is positive.
If x is positive and n is an odd positive integer, then $x^{1/n}$ is positive.
If x is negative and n is an even positive integer, then $x^{1/n}$ is not a real number.
If x is negative and n is an odd positive integer, then $x^{1/n}$ is negative.

## RULES OF RADICALS

Product Rule: If x and y are any positive real numbers and n is a natural number, then

$$\sqrt[n]{x} \cdot \sqrt[n]{y} = \sqrt[n]{xy} \quad \text{or} \quad x^{1/n} \cdot y^{1/n} = (xy)^{1/n}.$$

Note: This rule is frequently used when working with quantities with units. Suppose you wish to know the length of an edge of a square with an area of 25 $\text{in}^2$. The length is related to the area by, $L^2 = A$, so $L = \sqrt{A} = \sqrt{25\,\text{in}^2} = \sqrt{25} \cdot \sqrt{\text{in}^2} = 5\,\text{in}$.

Quotient Rule: If x and y are any positive real numbers ($y \neq 0$) and n is a natural number, then

$$\sqrt[n]{\frac{x}{y}} = \frac{\sqrt[n]{x}}{\sqrt[n]{y}} \quad \text{or} \quad \left(\frac{x}{y}\right)^{1/n} = \frac{x^{1/n}}{y^{1/n}}.$$

Power Rule: If m and n are positive integers, then

$$x^{m/n} = \left(x^{1/n}\right)^m = \left(x^m\right)^{1/n}$$

provided all indicated roots exist.

If $x^{m/n}$ exists and x ≠ 0, then $x^{-m/n} = \dfrac{1}{x^{m/n}}$ .

Since radicals can be written in fractional exponent form, these and other rules follow from the definitions and rules of exponents.

COMMONLY CONFUSED EXPRESSIONS

(1) $x^{1/n} \neq \dfrac{1}{x^n}$

(2) $\sqrt[n]{x+y} \neq \sqrt[n]{x} + \sqrt[n]{y}$

EXERCISES

1. Find examples that illustrate commonly confused expressions (1) and (2).

2. Without using a calculator, find the numerical equivalent of (that is, solve) each of the following.

a. $\sqrt{4} + \sqrt{25}$ b. $64^{1/2}$ - $36^{1/2}$ c. $\sqrt[3]{27} - \sqrt[4]{16}$

d. $125^{1/3}$ + $625^{1/4}$ e. $\sqrt[3]{8}\sqrt{9}$ f. $125^{1/3}$ x $25^{1/2}$

g. $\dfrac{\sqrt[3]{100\,\sqrt[3]{10^{12}}}}{\sqrt[3]{10^9}}$ h. $(10^{27})^{-1/3}$ x $(10^{10})^{2/5}$

3. Simplify each expression if possible. Assume that all indicated roots exist.

a. $\sqrt{a} + \sqrt{b}$ b. $\sqrt[3]{\sigma} - \sqrt[4]{\sigma}$

c. $2\sqrt[3]{\sigma} - \sqrt[3]{\sigma}$ d. $2\sqrt[3]{\sigma} \cdot \sqrt[3]{\sigma}$

e. $\left(\sqrt[3]{\left(27x^2y\right)^6}\right)\left(\sqrt[3]{\left(8x^3y^6\right)^2}\right)$ f. $\left(\sqrt[3]{\left(6x^2y\right)}\right)\left(\sqrt[3]{\left(36x^4y^2\right)}\right)$

g. $\sqrt[3]{(\alpha+\beta)^6}\sqrt{(\alpha+\beta)^6}$ h. $\dfrac{\left(\sqrt[3]{(\lambda+\mu)^6}\right)(\lambda+\mu)}{\sqrt[3]{(\lambda+\mu)^9}}$

i. $\dfrac{\sqrt[3]{(\lambda+\mu)^6}+(\lambda+\mu)}{\sqrt[3]{(\lambda+\mu)^9}}$

j. $\left[(z+T)^{36}\right]^{-1/3}\times\left[(z+T)^{27/6}\right]^2$

k. $\dfrac{\left(\sqrt[3]{(a+b)^5}\right)\left(\sqrt[3]{(a+b)^4}\right)}{\sqrt[3]{(a+b)^{12}}}$

l. $\dfrac{\left(\sqrt[5]{(ab)^{51}}\right)\left(\sqrt[5]{(ab)^{21}}\right)}{\sqrt[5]{(ab)^{12}}}$

m. $\dfrac{\left(\sqrt[3]{\left(3p^2q\right)^5}\right)\left(\sqrt[3]{(3p^5q^2)^4}\right)}{\sqrt[3]{(p^6q^2)^2}}$

n. $\dfrac{\left(\sqrt[4]{\left(6x^2y^3\right)^2}\right)\left(\sqrt[4]{(3x^3y^4)^5}\right)}{\sqrt[4]{(12x^5y^2)^3}}$

# REVIEW OF EXPONENTS AND ROOTS
## More Practice Exercises

Simplify each expression. Assume that all variables used as exponents represent integers and that all other variables represent nonzero real numbers. Write all answers with only positive exponents. Evaluate whenever possible.

1. $\left(\beta^3\alpha^2\right)^4\left(\beta^6\alpha^{-3}\right)^{-2}$

2. $\left(v^{-6}q^2\right)^4\left(v^6q^{-3}\right)^{-4}$

3. $(2^{-3})(3^{-2})$

4. $(-3c^2b^4)^3$

5. $\dfrac{12^7}{12^5}$

6. $\dfrac{m^3n^5}{m^8n^2}$

7. $\left(-2k^{-2}p^2\right)^{-2}\left(-3k^2p^{-3}\right)^{-1}$

8. $\dfrac{(-f^2)^3(2f^3)^{-2}\left(-3f^2\right)^{-1}}{(6f^2)^{-3}}$

9. $\dfrac{(-3ab^2)^3(2a^2b^{-3})^4}{(a^5b)^{-6}}$

10. $\dfrac{w^7+w^5-w^{-4}}{w^{-5}}$

11. $\dfrac{(\sigma+\delta)^5(\sigma+\delta)^{-7}}{(\sigma+\delta)^8(\sigma+\delta)^{-12}}$

12. $\dfrac{-5d^4(-2d^6g^{-2})^4}{(4d^3)^3 10g^{-10}}$

13. $16^{1/4}\left(8^{1/3}\right)$

14. $64^{1/2}\left(27^{1/3}\right)$

15. $\left(a^{16}b^3\right)^{1/4}\left(ab^{1/8}\right)^{3/2}$

16. $\dfrac{\sqrt[5]{x^9y^4}\,\sqrt[5]{x^6y}}{x^2}$

17. $\dfrac{\left(\sqrt[3]{r^7}\right)\left(\sqrt[3]{r^{20}}\right)}{r^3}$

18. $\dfrac{\sqrt[4]{x^6y^{12}}\,\sqrt{xy^3}}{xy^2}$

19. $\dfrac{k^{1/3}k^{1/4}}{k^{1/6}}$

20. $\dfrac{\left(\sqrt[4]{s^6r^7}\,\sqrt{rt^6}\right)^3}{\sqrt{s^3t^2}}$

# Algebraic Equations

### Types of Equations and Solutions

Solutions to Equations are values for the variable or variables that make the equation true.

Examples: x = 3 is a solution to the equation x + 2 = 5, because 3 + 2 = 5

x = 3 and y = 2 is a solution to the equation y = x – 1, because 2 = 3 – 1

x = 3 is a solution to the equation $x^2 = 9$, because $3^2 = 9$

x = -3 is also a solution to the equation $x^2 = 9$, because $(-3)^2 = 9$

Note: When we list multiple solutions to a single equation, we separate the solutions with the word or. Since $x^2 = 9$ is true if x = 3 or x = -3, we say that the equation $x^2 = 9$ is solved if x = 3 or x = -3. In this case we can save writing with the plus-or-minus-symbol, "±", and write the solutions to $x^2 = 9$ as x = ±3.

Not all equations have solutions. In this class, we will restrict ourselves to real solutions to equations. For example, $x^2 = -1$ has no real solutions. An equation with no solutions is called a contradiction. An example of a contradiction is the equation x = x + 2. There is no value you can substitute for x and have both sides of the equation be the same value.

Some equations are true for some, but not all, values of the variable or variables. Such an equation is called a conditional. An example of a conditional is x = x + y. This equation is true provided y = 0, but is not true if y is any other value.

Other equations are true for all values of the variable or variables. Such an equation is called an identity. With an identity, one side of the equation can be rearranged into the other side of the equation. Examples include distribution, 2(x - 3) = 2x - 6, obtaining a common denominator, $\frac{x}{2}+\frac{1}{6}=\frac{3x+1}{6}$, definitions, $a^3 = a \cdot a \cdot a$, and rules, $a^3 \cdot a^m = a^{3+m}$.

**A. Classify each of the following equations as an identity, a conditional or a contradiction. Assume all variables are positive real numbers.**

1. $\frac{A}{B}+\frac{C}{D}=\frac{AD+CB}{BD}$

2. $\frac{A}{C}+\frac{B}{D}=\frac{A+B}{C+D}$

3. $\frac{\lambda}{\alpha+\beta}=\frac{\lambda}{\alpha}+\frac{\lambda}{\beta}$

4. $\frac{\alpha+\beta}{\lambda}=\frac{\alpha}{\lambda}+\frac{\beta}{\lambda}$

5. $\frac{(G+Q)/(k+d)}{(k+d)/(G+Q)}=1$

6. $\frac{\sigma}{\theta/\phi}=\frac{\sigma\cdot\phi}{\theta}$

7. $\frac{\sigma}{\theta/\phi}=\frac{\sigma\cdot\theta}{\phi}$

8. $\frac{R}{\sigma}\times\frac{S+T}{\beta}=\frac{RS+RT}{\sigma\beta}$

9. $\frac{R}{\sigma}\div\frac{S+T}{\beta}=\frac{R\beta}{\sigma S+T}$

10. $\frac{1}{C_1}+\frac{1}{C_2}=\frac{C_1C_2}{C_1+C_2}$

## Solving an Equation for an Unknown Variable

Two equations with the same solutions are said to be equivalent. An example of a pair of equivalent equations is x + 2 = 5 and x = 3. Both of these equations are true only if x = 3.

Given any equation (x = 3), you can obtain an equivalent equation by doing any of the following:

Add the same thing to both sides of the equation, x + 2 = 3 + 2 or x + 2 = 5
Subtract the same thing from both sides of the equation, x - 1 = 3 - 1 or x - 1 = 2
Multiply both sides of the equation by the same thing (except 0!), 2·x = 2·3 or 2x = 6
Divide both sides of the equation by the same thing (except 0!), x÷3 = 3÷3 or $\frac{1}{3}$x = 1

Of course subtraction is the same as addition of a negative and division is the same as multiplication by a reciprocal, so we don't really need all of the examples, as long as we remember not to multiply or divide by zero.

The equations x = 3, x + 2 = 5, x - 1 = 2, 2x = 6, and $\frac{1}{3}$x = 1 are all equivalent, because they are true only for x = 3. That is to say, x = 3 is the only solution to each of these equations.

If we are asked to solve the equation x + 2 = 5 for x, we would subtract 2 from both sides and have x + 2 - 2 = 5 - 2 which simplifies to x = 3.

To solve 2x = 6 for x, we could multiply both sides by ½. (½)(2x) = (½)(6) which simplifies to x = 3. Instead of multiplying by ½, we could have divided by 2 and obtained the same result.

Sometimes to solve, we must perform more than one step. Suppose we wish to solve the equation $2a+b=12$ for $a$. This equation contains two variables, $a$ and $b$. When we solve for $a$, we are considering $b$ to be a known quantity. We may have a single value for $b$ or we may have a list of values, i.e. "Determine the value of $a$ when $b$ is 1, 2, 3, 4, and 5". (We will see in a later section that this can be useful when graphing the solutions for an equation with two variables, where we will solve an equation for one variable and plug in values for the other variable to determine ordered pairs to plot.)

$$2a+b=12$$
$$2a+b-b=12-b$$
$$2a=12-b$$
$$\tfrac{1}{2}\cdot 2a=\tfrac{1}{2}\cdot(12-b)$$
$$a=6-\tfrac{b}{2}$$

Not all solutions can be found with these simple steps. Sometimes we must use exponents or roots, the quadratic formula, or other methods to find a solution.

**Example Requiring a Root:** The solutions to the equation $x^2 = 9$ are found by taking the square root of each side of the equation. When we use a square root, we have to remember that we will have both a positive and a negative solution, x = ±3 in this case.

**Example Requiring an Exponent:** The solution to the equation $\sqrt{b}=9$ is found by squaring both sides of the equation. The result in this case is $b=81$.

**Quadratic formula:** The quadratic formula is frequently used to solve algebraic problems in physics.

When the quadratic formula is written as $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, it is the solution to the quadratic equation $ax^2 + bx + c = 0$ for the variable $x$. The equation $ax^2 + bx + c = 0$ contains 4 variables, but only $x$ is considered unknown.

Simple Examples: Solve $2x^2 - 5x + 2 = 0$ for $x$.

$$x = \frac{-(-5) \pm \sqrt{(-5)^2 - 4 \cdot 2 \cdot 2}}{2 \cdot 2} = \frac{5 \pm \sqrt{9}}{4} = \frac{5 \pm 3}{4} = \frac{8}{4} \text{ or } \frac{2}{4} = 2 \text{ or } \frac{1}{2}$$

Solve $4h^2 - 12h = 0$ for $h$.

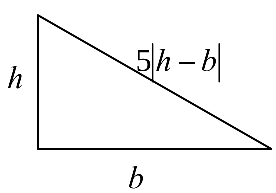
$$h = \frac{-(-12) \pm \sqrt{(-12)^2 - 4 \cdot 4 \cdot 0}}{2 \cdot 4} = \frac{12 \pm 12}{8} = \frac{24}{8} \text{ or } \frac{0}{8} = 3 \text{ or } 0$$

Notice that in the second example, "c" was zero in the quadratic formula because there was no constant term in the quadratic equation $4h^2 - 12h = 0$. When using the quadratic formula, "b" and/or "c" can be zero but "a" cannot. Why?

The coefficients "a", "b" and "c" are not always given in the original equation and can contain variables.

**More Complex Example requiring the Quadratic Formula:**

A right triangle has a base b, a height h and a hypotenuse that is 5 times the difference between the height and base. Determine the height of the triangle. We will learn in a later section the steps for setting up geometric word problems. In this case we will draw a picture and apply the Pythagorean Theorem.




$$b^2 + h^2 = (5|h - b|)^2$$
$$b^2 + h^2 = 25h^2 - 50hb + 25b^2$$
$$0 = 24h^2 - 50bh + 24b^2$$

We now apply the quadratic formula with 24 as "a", (-50b) as "b" and 24b$^2$ as "c".

$$h = \frac{-(-50b) \pm \sqrt{(-50b)^2 - 4 \cdot 24 \cdot 24b^2}}{2 \cdot 24} = \frac{50b \pm \sqrt{2^2 25^2 b^2 - 2^2 24^2 b^2}}{2 \cdot 24}$$
$$= \frac{2 \cdot 25b \pm \sqrt{2^2 b^2 (25^2 - 24^2)}}{2 \cdot 24} = \frac{2 \cdot 25b \pm 2b\sqrt{(25-24)(25+24)}}{2 \cdot 24}$$
$$= \frac{2b}{2 \cdot 24} \cdot \left(25 \pm \sqrt{(1)(49)}\right) = \frac{b}{24}(25 \pm 7) = \frac{b}{24}(32) \text{ or } \frac{b}{24}(18) = \frac{4}{3}b \text{ or } \frac{3}{4}b$$

So the height is $\frac{3}{4}b$ or $\frac{4}{3}b$.

**B. Solve for the specified unknown variable in each of the following examples:**

1. Consider $F$ to be unknown: $A + B - F = G + 2t$

2. Consider $t$ to be unknown: $A + B - F = G + 2t$

3. Consider $a$ to be unknown: $\dfrac{\theta + r}{a} - 2a = 0$

4. Consider $\lambda$ to be unknown: $\dfrac{\alpha + \beta}{\lambda} = \dfrac{\lambda}{\alpha + \beta}$

5. Consider $\alpha$ to be unknown: $\dfrac{2\alpha + \beta}{\lambda} = \dfrac{\alpha - \beta}{\lambda}$

6. Consider $d$ to be unknown: $\dfrac{b}{c + d} - 4cH = 3c$

7. More Challenging: Redo #6, considering $c$ to be unknown.

Actual Physics Examples: Here are some actual physics equations that you will likely run into. Notice that you can do the algebra without knowing what the equations mean! (You *will* need to understand the physics to *set up* the equations so that they are ready to solve.)

Notice that, in some cases, you do *not* have numbers to "plug in," while in others you do. In the latter case, however, it is usually advisable to rearrange the symbols before putting in the numerical values. This serves at least two purposes:

1. It makes it easier to find mistakes. If you put in the numbers early on, erroneous algebraic steps are not easy to find.

2. You will end up with a *general* solution. A general solution can be used to investigate questions like "What if the object started with twice the speed?", "What angle gives the maximum distance?", "What if there were no friction?", and so on. A general solution is also useful in lab situations.

Be ready to use the two algebraic quick steps, to group terms and factor, to raise both sides to a power or take roots of both sides, and to use the quadratic equation.

**C. Solve for the specified unknown variable in the following physics equations.** Assume that all other variables are known. If numerical values are given put them in at the end to obtain a numerical result. [Note: In these cases the known variables have a numerical value and units. Simplify the units as you would variables using the rules of exponents and roots.]

1. $t$ = ?: $v_1$ = 30 m/s ,$v_2$ = 10 m/s, $a$ = –10 m/s$^2$ : $v_2 = v_1 + at$

2. $T_1$ = ?: $Q = c\,m\,(T_2 - T_1)$

3. $r$ = ? : $q\,v\,B = m\dfrac{v^2}{r}$

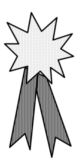
4. $L$ = ?, $g$ = 9.8 m/s$^2$, $T$ = 1.0 s : $T = 2\pi\sqrt{\dfrac{L}{g}}$

5. $R_{eq}$ = ?, $R_1$ = 2 Ω, $R_2$ = 4 Ω: $\dfrac{1}{R_1} + \dfrac{1}{R_2} = \dfrac{1}{R_{eq}}$

6. $m$ = ?, $F$ = 20.0 N, $T$ = 4.0 N, $\mu_k$ = 0.15, $g$ = 9.8 m/s$^2$, a = 3.0 m/s$^2$: $F - \mu_k\,m\,g + 0.866\,T = m\,a$

7. $r$ = ? : $\dfrac{1}{2}m\,v^2 - G\dfrac{M\,m}{r} = \dfrac{1}{2}k\,x^2$

8. $t$ = ?, $\Delta y$ = 5.0 m, $v_1$ = 20 m/s, $g$ = 10 m/s$^2$: $\Delta y = v_1 t - \frac{1}{2}gt^2$



If you can work through these fluently (accurately and quickly), you are well on your way towards mathematical preparedness for physics. If not, you can get there. Just practice, practice, practice until you get the game.

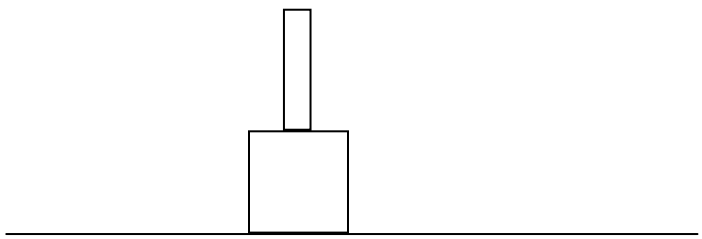
# Preparation for Problem Solving I

Before beginning problem solving (word problems) it is necessary to develop some basic skills to be applied to the analysis of problems. First of all, one must be able to read the statement of a problem and translate it into symbols and pictures. These exercises are designed to develop these skills.

## Words to Pictures

Sketch a diagram to represent each situation described.

Example: A rectangular block is placed with its smallest side on top of a cube.



(1) A ladder is leaned against a wall so that its top is twice as far from the floor as its bottom is from the wall.

(2) A chair is tipped back on its rear legs so that they make an angle ø with respect to the floor.

(3) A rectangle has a length that is twice its width.

(4) A small ball is resting on top of a large box that is sitting on a table.

(5) Three blocks are stacked on an inclined plane with the smallest block in the middle and the largest block on top.

(6) Two train stations are located 200 miles apart. One train leaves station A heading toward station B at the same time a second train leaves station B heading toward station A. The two trains pass each other 50 miles from station A.

(7) A 10-kg block and a 5-kg block are joined by a string that passes over a pulley hung from the ceiling. The 5-kg block is hanging lower than the 10-kg block.

(8) A flat board is sitting on top of a rectangular box. A second box is resting on top of the board.

## Words to Algebra

Assign a symbol to represent each quantity. Convert the sentence into an algebraic expression (equation).

Example: the density of an object is defined to be its mass divided by its volume.

Let $\rho$ = density, m = mass, and V = volume,

then $\rho = m/V$.

(1) The temperature expressed in kelvin is 273.15 greater than the temperature expressed in degrees celsius.

(2) The average of three distances is the sum of the distances divided by three.

(3) The distance traveled by a car moving at a constant speed is the product of its speed and the time of travel.

(4) The area of a triangle is one-half the product of its base and its height.

(5) Object A is five times as long as object B.

(6) If Mary were 30 pounds heavier she would be twice as heavy as Eric.

(7) An object starts from rest and travels a certain distance in a straight line with a constant acceleration. The time this takes is the square root of the quantity twice the distance divided by the acceleration.

(8) The friction force acting on a block sliding down an incline is one-tenth of the weight of the block.

(9) If the length of block 1 were tripled it would be equal to half the length of block 2.

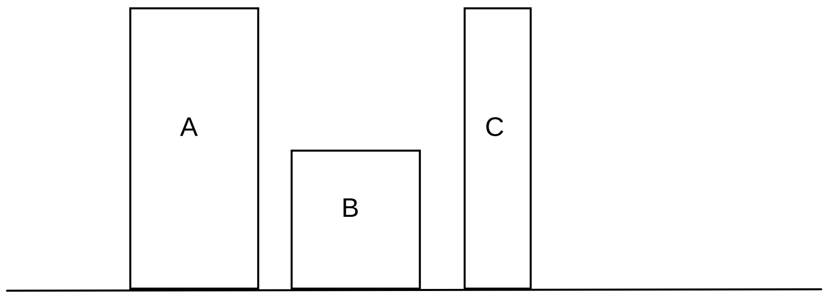
(10) The length of a rectangle is eight inches more than triple its width.

# Preparation for Problem Solving II

Another important skill in doing physics problems is to be able to visualize the physical process being analyzed. It is difficult to construct diagrams or algebraic relations if you cannot visualize what's going on. These simple set-ups will provide a foundation for visualizing both mechanical systems as well as models for non-mechanical systems.

## Words to Set-Ups

Use the set of objects provided by your instructor to physically set-up what is described here. Work in groups of at least two and make sure everyone in you group agrees with the final set-up. Have your instructor check that your set-up precisely matches the description.




(1) A large rectangular block is placed on the table with its smallest face in contact with the table. An identical block is placed on top of the first block with the second block's largest face on top of the first block. [How many correct variations of this set-up can you come up with?]

(2) Three blocks are stacked on an inclined plane with the smallest block in the middle and the largest block on top.

(3) Place a type C-block so that one of its largest faces is in contact with the table. Place an A-block on the table with its smallest face on the table. Place the two blocks such that the smallest face of the C block is two lengths (longest dimension) of the large block away from the second largest face of the large block.

(4) A B-block is placed with its largest face on an inclined surface. A C-block is placed on top of the B-block with its smallest face touching the B-block. The ramp is then inclined to the maximum angle that can be achieved without the C-block toppling.

(5) A C-block is placed on top of a B-block so that it lies across the diagonal of the largest face of the B-block. The C-block is centered on this diagonal. An A-block is placed small-face-down on top of the C-block and is rotated so as to make a right angle with the C-block. [How many right angles does the C-block make with the A-block?]

(6) An A-block is on the table with its largest side down. Place a B-block on its largest side so that its smallest side is touching the smallest side of the A-block.

(7) Place 2 C-blocks on the inclined plane, with the smallest faces touching the inclined plane. The two C-blocks should be separated by a distance equal to the longest side of a B-block. Place an A-block on top of the C-blocks with the largest face of the A-block touching the tops of the C-blocks.

The next two exercises have you construct models that show key states of physical processes.

(8) Two trains are located 200 miles apart. One train (use a single A-block) leaves station A heading toward station B at the same time a second train (use a B-block) leaves station B heading toward station A. The two trains pass each other 50 miles from station A. Using two A-blocks and two B-blocks, construct a physical model that shows both states (instants) described.

(9) A California condor flying over the desert admires the beautiful hues and shading of the soft dusk light. She looks down and sees telephone poles (C-blocks) on a straight, horizontal road. From experience, the condor knows that the distance between the poles is equal to the height of one pole. She spots two cars (use B-blocks) approaching each other with constant speeds. One car is at pole #1, the other is at pole #7. She notices that the car at pole #1 is traveling twice as fast as the car at pole #7, so—without a calculator!—she quickly predicts where the cars will pass. Place two B-blocks in the observed positions. Leave them there and place two more B-blocks in the predicted passing positions.

Now it's your turn!

(10) Write (in words) the description of three different challenge set-ups for your partner(s). One description should be of unstacked blocks on a horizontal surface—describe distances and orientations. The second description should be of blocks stacked on a horizontal surface. The third description should be of blocks stacked on an incline. Exchange written descriptions with your partner(s). Construct the set-ups that your partner has described (on paper) for you.

# Review of Exponential and Logarithmic Functions

Definition: For $b > 0$ and $b \neq 1$, and all real numbers x,

$$y = b^x$$

is an exponential function with "base b."

Property: For $b > 0$ and $b \neq 1$, if $b^x = b^y$, then $x = y$.

Definition: For $b > 0$ and $b \neq 1$, and all positive numbers x,

$$y = \log_b x$$

means the same as $x = b^y$.

Properties: If $b > 0$ and $b \neq 1$, then $\log_b b^x = x$.

If $b > 0$ and $b \neq 1$ and $x > 0$, then $b^{\log_b x} = x$.

For $b > 0$ and $b \neq 1$, $\log_b b = 1$ and $\log_b 1 = 0$.

Rules of Logarithms

Product Rule: If x, y, and b are positive numbers, $b \neq 1$, then

$$\log_b xy = \log_b x + \log_b y.$$

Quotient Rule: If x, y, and b are positive numbers, $b \neq 1$, then

$$\log_b (x/y) = \log_b x - \log_b y.$$

Power Rule: If x and b are positive real numbers, $b \neq 1$, and if n is any real number, then

$$\log_b x^n = n \log_b x.$$

Verifying the rules of logarithms.

To prove the product rule, rewrite the logarithmic statements in exponential form

$\log_b x = m$ means $b^m = x$ and $\log_b y = n$ means $b^n = y$.

So, by substitution, and using the product rule for exponents,

$$xy = b^m \cdot b^n = b^{m+n}.$$

Rewrite this in logarithmic form,

$$\log_b xy = m + n.$$

Now substitute for m and n to get

$$\log_b xy = \log_b x + \log_b y.$$

These rules and properties can be used to solve algebra problems as shown in the following example.

Example: Solve the following equation for x: $5^{x+3} = 25$.

Take $\log_5$ of both sides of the equation: $\log_5 5^{x+3} = \log_5 25$

This leads to $x + 3 = \log_5 5^2$ which becomes $x + 3 = 2$. Solving gives $x = -1$.

Commonly Confused Expressions

(1) $\log_b (x \pm y)$ is not the same as $\log_b x \pm \log_b y$.

(2) $\log_b (xy)$ is not the same as $(\log_b x)(\log_b y$.)

EXERCISES

1. a. For each statement given in exponential form, rewrite it in logarithmic form.

   i. $2^x = 4$ ii. $10^5 = 100{,}000$ iii. $4^{-3} = 1/64$

   b. For each statement given in logarithmic form, rewrite it in exponential form.

   i. $\log_2 8 = 3$ ii. $\log_{10} 1{,}000{,}000 = 6$ iii. $\log_3 \frac{1}{9} = -2$.

2. Solve the following equations for the given variable.

   a. $2^x = 4$ b. $9^x = 27$ c. $4^x = 64$ d. $3^x = \frac{1}{9}$

   e. $\left(\frac{1}{2}\right)^x = 8$ f. $6^{-x} = \frac{1}{6}$ g. $2^{3x+2} = 16$ h. $16^{(x+1)/2} = 32$

*Problems 3 and 4 may be done as in-class exercises. Check with your instructor.*

3. Show why for the first property (For $b > 0$ and $b \neq 1$, if $b^x = b^y$, then $x = y$.), the statement "$b \neq 1$" is necessary.

4. Verify the quotient rule and the power rule.

5. Evaluate the following.

   a. $\log_{10} 1{,}000$ b. $\log_{10} 0.001$ c. $\log_{10} 1$ d. $\log_8 64$

   e. $\log_5 125$ f. $\log_3 (1/27)$ g. $\log_5 25^2$ h. $\log_3 27^{1/2}$

6. Solve each equation for the given variable.

   a. $y = \log_6 216$. b. $\log_5 x = -3$ c. $\log_x 9 = 1/2$ d. $\log_c 125 = -3$

   e. $\log_4 x = 5/2$ f. $\log_2 \mu = 0$ g. $\log_p 0.1 = 1$ h. $\log_{1/2} f = 1$

7. Write each of the following as a single logarithm. Assume all variables represent positive real numbers. Simplify as much as possible.

   a. $\log_6 2 + \log_6 3$ b. $\log_{10} 50 - \log_{10} 5$ c. $3\log_5 2 + \log_5 6^2$

8. Find examples that illustrate the commonly confused expressions.

# Applications

Physics requires the application of math skills to solve problems. In these exercises, we will apply the math concepts reviewed in this section to some problems.

## Geometric Applications

Some problems encountered in the real world require the knowledge of simple geometric formulas.

Example 1 Suppose you are asked to determine the width of a rectangle which has a length which is twice its width and has a perimeter of 18 feet.

To solve this we will follow a set of problem solving steps.

1. **Draw a picture**

   This problem is about a rectangle, so we draw a rectangle:

2. **Assign a symbol to each known and unknown quantity and label your diagram with those**

   Given: P = perimeter of rectangle = 18 ft
   Unknown: w = width of rectangle and L = 2w = length of rectangle


3. **Determine the relationship between the known and unknown quantities**

   $P = 2L + 2w = 2*2w+2*w$ or $P = 2w + w + 2w + w$

4. **Solve symbolically**

   $P = 6w \rightarrow w = P/6$

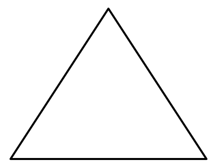
5. **Substitute known values and evaluate numerically**

   $w = P/6 = (18\text{ ft})/6 = 3\text{ ft}$

This very simple example illustrates our problem solving steps.

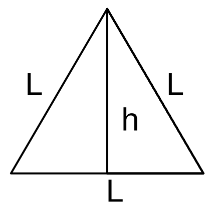
Example 2 Determine the area of an equilateral triangle with sides of length L.

**1. Draw a picture**



**2. Assign a symbol to each known and unknown quantity and label your diagram with those**

Given: L = length of each side; b = base of triangle = L
Unknown: h = height of triangle
Unknown: A = area of triangle




**3. Determine the relationship between the known and unknown quantities**

Considering the right triangle on the right with hypotenuse L and legs h and ½ L: $h^2 + (\frac{1}{2} L)^2 = L^2$.

$A = \frac{1}{2} b \cdot h = \frac{1}{2} L \cdot h$

**4. Solve symbolically**

Find h: $h^2 + \frac{1}{4} L^2 = L^2 \rightarrow h^2 = \frac{3}{4} L^2 \rightarrow h = \sqrt{\frac{3L^2}{4}} = \frac{\sqrt{3}\,L}{2}$

Compute A: $A = \frac{1}{2} L \cdot h = \frac{1}{2} L \cdot \frac{\sqrt{3}\,L}{2} = \frac{\sqrt{3}\,L^2}{4}$

**5. Substitute known values and evaluate numerically**

In this case there is no numerical value to substitute for L.
$A = \frac{\sqrt{3}\,L^2}{4}$ is the final result.

## Exponential Growth and Decay Applications

In many areas of science as well as other fields, the growth rate of a quantity is proportional to its current value. For example, the growth rate of a population of buffalo is proportional to the number of buffalo present in the absence of any predators. In such a case, there is an exponential growth.

Example 3 Suppose the population of a herd of buffalo in Montana doubles every 3 years. Today, there are 20 buffalo in the herd. How many years will it take before there are 640 buffalo in the herd?

There is no picture to draw in solving this problem. So, we will begin our process by assigning symbols to the unknown and known quantities. Our problem solving steps are as follows:

**1. Assign a symbol to each known and unknown quantity**

Unknown: t = time for population to reach 640
Given: $t_d$ = time for population to double = 3 yr
Given: $N_0$ = number of buffalo present today = 20
Given: N = number of buffalo present at time t = 640

**2. Determine the relation between the known and unknown quantities**

$$N = N_0 2^{t/t_d} = N_0 2^{\frac{t}{t_d}}$$

**3. Solve symbolically**

$$N = N_0 2^{t/t_d} \rightarrow 2^{t/t_d} = \frac{N}{N_0} \rightarrow \frac{t}{t_d} = \log_2\left(\frac{N}{N_0}\right) \rightarrow t = t_d \log_2\left(\frac{N}{N_0}\right)$$

**4. Substitute known values and evaluate numerically**

$$t = t_d \log_2\left(\frac{N}{N_0}\right) = (3\,\text{yr})\log_2\left(\frac{640}{20}\right) = (3\,\text{yr})\log_2(32) = (3\,\text{yr})(5) = 15\,\text{yr}$$

Many processes in science also involve exponential decay. Many chemical reactions have rates that are proportional to the amount of reactant present. For example, the decomposition of caffeine in the body occurs with a half-life of 6 hours. This means if you have 300 mg of caffeine present in your body at 8 am, you will have 150 mg present 6 hours later at 2 pm and 75 mg present after another 6 hours at 8 pm and so on.

Example 4 Suppose you take a No-Doze pill which contains 200 mg of caffeine. How long will it take for the amount of caffeine in your body to drop to 25 mg?

**1. Assign a symbol to each known and unknown quantity**

Unknown: t = time for amount of caffeine to reach 25 mg
Given: $T_{1/2}$ = half-life of caffeine = 6 h
Given: $m_0$ = beginning amount (mass) of caffeine = 200 mg
Given: m = amount (mass) of caffeine present at time t = 25 mg

**2. Determine the relation between the known and unknown quantities**

$$m = m_0\left(\frac{1}{2}\right)^{\frac{t}{T_{1/2}}}$$

**3. Solve symbolically**

$$m = m_0\left(\frac{1}{2}\right)^{\frac{t}{T_{1/2}}} \rightarrow 2^{-\frac{t}{T_{1/2}}} = \frac{m}{m_0} \rightarrow -\frac{t}{T_{1/2}} = \log_2\left(\frac{m}{m_0}\right) \rightarrow t = -T_{1/2}\log_2\left(\frac{m}{m_0}\right)$$

**4. Substitute known values and evaluate numerically**

$$t = -T_{1/2}\log_2\left(\frac{m}{m_0}\right) = -(6\,h)\log_2\left(\frac{25\mu g}{200\mu g}\right) = -(6\,h)\log_2\left(\frac{1}{8}\right) = -(6\,h)(-3) = 18\,h$$

Exercises

1. Determine the area of a square, which has a diagonal with length 3 m.

2. A rectangle has a length which is 3 feet less than twice its width. The area of the rectangle is 14 $ft^2$ (14 square feet). What is the length of the rectangle?

3. The population of rabbits in a county is found to triple every 4 years. If there are currently 100 rabbits in the county and this trend continues, how many rabbits will there be in 16 years?

4. Epinephrine (commonly called adrenaline) is a hormone released from the adrenal glands in emergency situations or when danger is sensed. Epinephrine has a half life of 2 minutes. If your body releases 400 micrograms of epinephrine when a deer jumps in front of the car you are riding in, how long will it take for the amount of epinephrine in your body to drop to 25 micrograms?

# Review #1: Basic Math

**All exercises should be done without calculators.**

A. Simplify. Assume all variables represent non-zero real numbers. Write all answers with only positive exponents.

1. $(3^{-2})^3$
2. $(-2^2)^3$
3. $(2/3)^{-3}$
4. $6^{12} \cdot 6^{10}$
5. $-3^2 \cdot 2^3$
6. $\dfrac{-5^{-2}}{5^{-3}}$
7. $m^6 \cdot m^{-8}$
8. $f^{\,x} \cdot f^{3x}$
9. $\dfrac{a^{-2}a^4}{a^6a^{-13}}$
10. $5(-2b)^2\,(b^3)^6$
11. $\dfrac{-3^4\,\omega^2\rho^{-3}}{(-6)^3\,\omega^{-8}\rho^5}$
12. $\dfrac{(\sin\phi)^2(\sin\phi)^{-8}}{(\sin\phi)^6}$
13. $\dfrac{(x+y)^3(x+y)^2(x+y)^{-8}}{(x+y)^5(x+y)^{-16}}$
14. $\dfrac{\left(a^3b^{-2}c^5\right)^{-4}\left(-2a^2b^4c\right)^3\left(-3ac^{-2}\right)^{-2}}{\left(2a^6b^3\right)^4\left(-b^{-3}c^5\right)^{-2}}$
15. $[(\psi+\lambda)^2(\psi+\lambda)^{-3}(\psi+\lambda)^{15}+(\psi+\lambda)^8(\psi+\lambda)^2]^0$

B. Simplify. Assume all variables represent positive real numbers.

1. $8^{2/3}$
2. $\sqrt{3.6\times10^5}$
3. $5^{1/4}5^{7/4}$
4. $\sqrt[3]{\left(3 \text{ x } 10^2\right)\left(9 \text{ x } 10^{10}\right)}$
5. $-2^2 \cdot 3^3$
6. $\sqrt[3]{-64}$
7. $\sqrt[5]{\dfrac{(x+y)^2}{(x+y)^7}}$
8. $\sqrt{\dfrac{36}{81}}$
9. $\sqrt[5]{-m^{10}n^{25}}$
10. $\sqrt[4]{\left(100^2\right)\left(10^6\right)\left(1000^2\right)}$
11. $\dfrac{r^{\frac{1}{4}}r^{\frac{1}{2}}}{r^{\frac{3}{4}}}$
12. $\left(\dfrac{z^{-1}x^{-3/5}}{2^{-2}z^{-1/2}x}\right)^{-5}$
13. $\dfrac{\sqrt[3]{\sigma\mu^2}\,\sqrt[3]{\sigma^5\mu^{-2}}}{\sqrt[3]{\sigma^8\mu^4}\,\sqrt[3]{\sigma^7\mu^5}}$
14. $3\sqrt{8}-5\sqrt{2}$

C. Rewrite each relation using logarithmic notation.

1. $x^3 = 4$
2. $2^{-5} = \dfrac{1}{32}$
3. $b^p = E$
4. $4^x = 64$

D. Rewrite each relation in exponential form.

1. $\log_b 3 = E$ 2. $\log_{10} 1000 = 3$ 3. $\log_3 x = y$ 4. $\log_8(1/8) = -1$

E. Write each of the following as a single logarithm. Assume all variables represent positive real numbers. Simplify as much as possible.

1. $\log_b 3\mu + \log_b 7$ 2. $\log_4 2 + \log_4 8$ 3. $6\log_x m - 2\log_x q$ 4. $x\log_b 3 - 3\log_b 3^x$

F. Solve each equation for the indicated variable. Simplify your results.

1. Solve for $\mu : 3\mu + \Delta = 3\Delta + \mu$

2. Solve for q: $\dfrac{1}{f} = \dfrac{1}{p} + \dfrac{1}{q}$

3. Solve for $t : x = vt + \frac{1}{2}at^2$

4. Solve for $\beta : \beta - 3\phi + 90 = 5\beta + 4\phi + 180$

5. Solve for $v : mgh = \frac{1}{2}mv^2 + mgy$

6. Solve for a: $Mg - Ma = \mu mg + ma$.

7. i. Solve for t: $x_f = x_i + v_i t + ½ a t^2$
 ii. Find t when $x_i = 0$ m, $x_f = 18.75$ m, $v_i = 0$ m/s, and $a = 1.5$ m/s$^2$.
 iii. Find t when $x_i = 18.75$ m, $x_f = 32.25$ m, $v_i = 7.5$ m/s, and $a = - 2.0$ m/s$^2$.

## Building Physical Intuition - Pushing Blocks

This exercise is designed for you to develop your ability to record and communicate observations you make in an effective manner.

Students should work in pairs.

Equipment: Each pair of students will need a wooden block and a writing instrument with an eraser for pushing the block.

Each group member should record all observations in ink on separate notebook paper. You will hand in this paper at the end of the activity.

1. Place the block on the table with its largest face down.

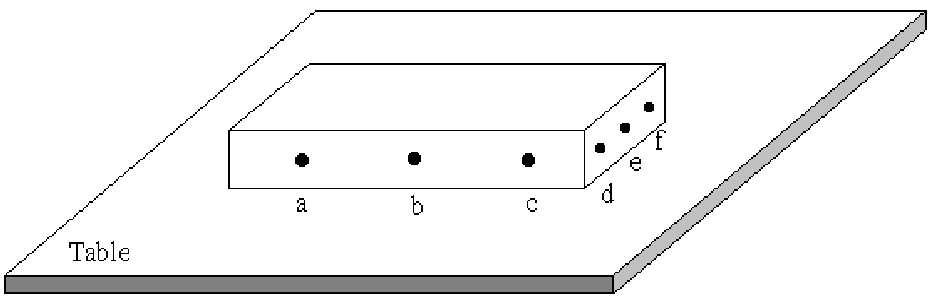



fig. 1

Place the eraser of your pencil at the position labeled "a" with the pencil parallel to the table and gently push. Observe the motion of the block. Record your observation using a complete sentence.

Repeat for each position (“b” through “f”) on the block.

2. Now place the block on the table as shown in figure 2.

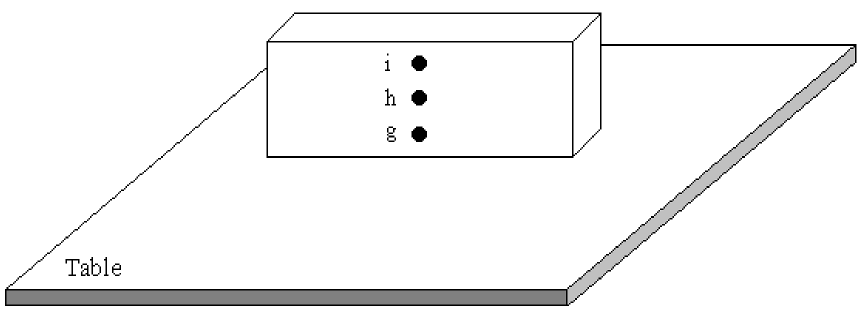



fig. 2

For each point, “g”, “h” and “i”, place the eraser of your pencil at the position with the pencil parallel to the table and gently push. Observe the motion of the block. Record your observation using a complete sentence.

Repeat the process with a harder push. Does the motion of the block depend on how gently (slowly) or how hard (quickly) you push it? Describe any differences.

# Building Physical Intuition – Stacked Blocks

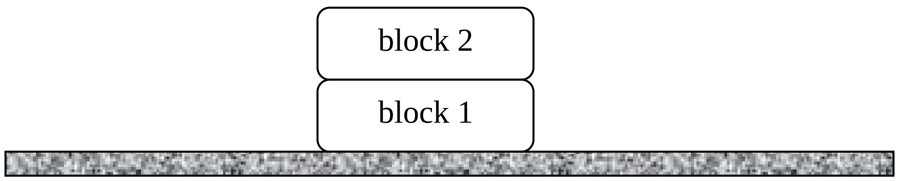
The goals of this activity are to continue to develop your observation skills and to begin to become aware of forces (pushes and pulls) and their effects on objects. You will make predictions, test your predictions, and explain differences between your predictions and your observations. Use your results to make predictions about new situations.

Students should work in pairs.

Equipment: Each pair of students will need five wooden blocks and a writing instrument with an eraser for pushing a block.

Each group member should record all predictions and observations in ink.

**Part A. Stack two blocks on the table as shown. Call the bottom block "block 1" and the top block "block 2".**




1. Predict what will happen when you push softly/slowly on the end of block 2.

Push softly/slowly on the end of block 2. Record your observations below.

Did your observations agree with your predictions? If they did not agree, can you offer an explanation?

2. Predict what will happen when you push softly/slowly on the end of block 1.

Push softly/slowly on the end of block 1. Record your observations below.

Did your observations agree with your predictions? If they did not agree, can you offer an explanation?

3. Predict what will happen when you push hard/fast on the end of block 1.

Push hard/fast on the end of block 1. Record your observations below.

Did your observations agree with your predictions? If they did not agree, can you offer an explanation?

Did block 2 move forward or backward relative to block 1?

Did block 2 move forward or backward relative to the table?

4. Discuss the differences in the results of 1, 2, and 3. Explain why these differences occurred.

**Part B. Stack a third block (block 3) on top of block 2.**

1. Predict what will happen when you push softly/slowly on block 2.

Push softly/slowly on the end of block 2. Record your observations below.

Did your observations agree with your predictions? If they did not agree, can you offer an explanation?

2. Predict what will happen when you push hard/fast on block 2.

Push hard/fast on the end of block 2. Record your observations below.

Did your observations agree with your predictions? If they did not agree, can you offer an explanation?

**Part C. Stack two additional blocks on top of blocks 1, 2, and 3. Feel the difference in the forces required to slide different blocks. For example, how much harder or easier is it to slide block 2 than to slide block 5 (top block)? Hypothesize what factors are responsible for your results.**

Record your observations, hypotheses, and any additional comments.

**Part D. If time permits, develop some additional experiments. Make predictions and test them.**

Record your observations, hypotheses, and any additional comments on a separate paper.